THE ABSTRACT GROUPS $G^{m,n,p}$*

BY

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CONTENTS

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PREFACE

This paper treats of three families of abstract groups, defined by the following sets of relations:*

$$(l, m \mid n, k): \quad R^l = S^m = (RS)^n = (R^{-1}S)^k = 1;$$

$$(l, m, n; q): \quad R^l = S^m = (RS)^n = (R^{-1}S^{-1}RS)^q = 1;$$

$$G^{m,n,p}: \quad A^m = B^n = C^p = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1.$$

Here, "$R^l=1$" means "R is of period l." When this and the remaining relations imply $R^{l/d}=1$, $(d>1)$, we acknowledge a case of *collapse*: *total* collapse if the relations reduce every generator to the identity, otherwise *partial*.

These groups are interrelated in various ways, of which the most important are as follows: $(m, m \mid n, k)$ is a subgroup of index two in $(2, m, 2n; k)$; $(2, m, n; q)$ is a subgroup of index two in $G^{m,n,2q}$.

The study of these families is justified by the number of important groups which they contain. In particular, the families include the following *simple groups*:

* The symbols are designed to admit the combination $(l, m \mid n, k; q)$ (symmetrical between l and m, and between n and k), which would mean the group defined by $R^l = S^m = (RS)^n = (R^{-1}S)^k = (R^{-1}S^{-1}RS)^q = 1$.

$\mathfrak{P}_1(5)$ (order 60) $\sim(5, 5 \mid 2, 3) \sim (2, 3 \mid 5, 5) \sim (2, 3, 5; 5) \sim G^{3,5,5}$,
$\mathfrak{P}_1(7)$ (order 168) $\sim(3, 3 \mid 4, 4) \sim (4, 7 \mid 2, 3) \sim (2, 3, 7; 4)$,
$\mathfrak{P}_1(9)$ (order 360) $\sim(5, 5 \mid 2, 4) \sim (3, 4, 5; 2)$,
$\mathfrak{P}_1(8)$ (order 504) $\sim G^{3,7,9}$,
$\mathfrak{P}_1(11)$ (order 660) $\sim G^{5,5,5}$,
$\mathfrak{P}_1(13)$ (order 1092) $\sim(l, 7 \mid 2, 3) \sim (2, 3, 7; l)$, $(l=6$ or $7)$,
$\mathfrak{P}_1(17)$ (order 2448) $\sim(4, 9 \mid 2, 3)$,
$\mathfrak{P}_1(19)$ (order 3420) $\sim(2, 5, 9; 2) \sim G^{3,9,9}$,
$\mathfrak{P}_1(23)$ (order 6072) $\sim(2, 3, 11; 4)$,
$\mathfrak{P}_1(29)$ (order 12180) $\sim G^{3,7,15}$.

Following Schreier and van der Waerden,* I let $\mathfrak{P}_1(q)$ denote the group commonly called $LF(2, q)$ or $PSL(2, q)$, q being a prime or a prime power. When q is odd, I let $\tilde{\mathfrak{P}}_1(q)$ denote the group of *all* linear fractional transformations in the Galois field of order q, otherwise called $PGL(2, q)$; this contains $\mathfrak{P}_1(q)$ as a subgroup of index two. The groups $\mathfrak{P}_1(9)$ and $\tilde{\mathfrak{P}}_1(9)$, being the alternating and symmetric groups of degree six, will usually be denoted by $G_{6!/2}$ and $G_{6!}$.

The above definitions for $\mathfrak{P}_1(5)$ (or $G_{5!/2}$) are merely redundant forms of Hamilton's definition†

$$\iota^2 = \kappa^3 = (\iota\kappa)^5 = 1.$$

The first definition for $\mathfrak{P}_1(7)$ is due to G. A. Miller; the second is immediately deducible from Burnside's; the third is essentially Dyck's‡ (but was put into precisely this form by H. R. Brahana§). The second definition for $\mathfrak{P}_1(13)$ is due to Brahana§ $(l=6)$ and Sinkov|| $(l=7)$; the first is easily deduced from it. The first definition for $\mathfrak{P}_1(9)$ (or $G_{6!/2}$), and the definitions for $\mathfrak{P}_1(8)$ and $\mathfrak{P}_1(17)$, have already been published;¶ but $(3, 4, 5; 2)$, $G^{5,5,5}$, $(2, 5, 9; 2)$, $G^{3,9,9}$, $(2, 3, 11; 4)$, $G^{3,7,15}$ are quite new.

We shall see (Theorem G) that, for every prime p, the linear fractional group $\mathfrak{P}_1(p)$ is a factor group either of some $(2, 3, p; q)$ or of some $G^{3,n,p}$. This result may be compared with the following.** For the group $\mathfrak{P}_1(2^m)$, every pair of generators of periods two and three satisfies the definition of some $G^{3,n,p}$.

* Schreier and van der Waerden [1]. Their Theorem 1 shows that $\tilde{\mathfrak{P}}_1(p)$ is the group of isomorphisms of $\mathfrak{P}_1(p)$ (p prime). We shall make frequent use of this theorem.

† Hamilton [1].

‡ Miller [1], p. 364; Burnside [1], p. 422; Dyck [1], p. 41.

§ Brahana [2], pp. 351, 354.

|| Sinkov [1], p. 239.

¶ Todd and Coxeter [1], p. 31; Sinkov [3], p. 70; Coxeter [7], p. 56.

** Sinkov [6], p. 454.

The following seven theorems will be proved:

THEOREM A. *The groups* $(2, m|n, k)$ *for* $n \neq k$, $(3, m|2, k)$ *for* $m \neq k$, $(5, m|2, 3)$ *for* $m \neq 5$, *and* $(l, m|2, 2)$ *for* $l > 2$ *and* m *odd, all collapse. Apart from these cases, if* l *and* m *are even, or if* l *and* k *are even and* $n = 2$, *the group* $(l, m|n, k)$ *is finite when* $2 \sin \pi/l \sin \pi/m > \cos \pi/n + \cos \pi/k$, *and infinite otherwise.*

THEOREM B. *For all infinite groups* $(l, m|n, k)$,

$$2 \sin \pi/l \sin \pi/m \leqq \cos \pi/n + \cos \pi/k.$$

THEOREM C. *If* $q > 1$ *and* $1/m + 1/n \leqq 1/2$, *and* m *and* n *are either even or equal (or both), the group* $(2, m, n; q)$ *is finite when*

$$\cos 2\pi/m + \cos 2\pi/n + \cos \pi/q < 1,$$

and infinite otherwise.

THEOREM D. *For all infinite groups* $(2, m, n; q)$,

$$\cos 2\pi/m + \cos 2\pi/n + \cos \pi/q \geqq 1.$$

THEOREM E. *If the smallest of* m, n, p *is greater than* 2, *while the next is greater than* 3, *and if these three numbers are either all even, or one even and the other two equal, the group* $G^{m,n,p}$ *is finite when*

$$\cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p < 1,$$

and infinite otherwise.

THEOREM F. *For all (noncollapsing) groups* $G^{m,n,p}$, *save* $G^{2,n,2n}$ *(n odd),*

$$(1 - \cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m - \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m + \cos 2\pi/n - \cos 2\pi/p) \geqq 0;$$

and for all infinite groups $G^{m,n,p}$,

$$\cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p \geqq 1.$$

THEOREM G. *If* p *is a prime, congruent to* 1 *or* 3 (mod 4), *the group* $\mathfrak{P}_1(p)$ *or* $\bar{\mathfrak{P}}_1(p)$ *(respectively) is a factor group of* $G^{3,n,p}$, *where* n *is the ordinal of the first Fibonacci number that is divisible by* p. *When* $p \equiv 3$ (mod 4), $\bar{\mathfrak{P}}_1(p)$ *is a factor group of* $(2, 3, p; n/2)$.

In the final section, we see that $G^{m,n,p}$ has an elegant representation on a "semi-regular map" of m-gons and n-gons (or n-gons and p-gons, or p-gons and m-gons). Although the representation has not (so far) given any new information about the groups, it provides an alternative method for enumerating the operators, whenever the order does not greatly exceed five hundred.

Moreover, the representation seems to clarify the phenomenon of "collapse."

Tables I, II, III, at the end of the paper, summarize the special results, and at the same time serve as an index. In Table III, it is noteworthy that the group $G^{3,7,p}$, with $p=12$ or 13 or 14, has order $12\times 13\times 14$. (Although $G^{3,7,12}\sim G^{3,7,14}$, $G^{3,7,13}$ is a different group of the same order.)

A considerable part of this work is the fruit of discussions with Dr. A. Sinkov, extending over several years. To him I would express my sincere gratitude.

Chapter I. $(l, m\,|\,n, k)$

1.1. **Introduction; the polyhedral groups.** The group $(l, m\,|\,n, k)$, which is defined by

$$R^l = S^m = (RS)^n = (R^{-1}S)^k = 1, \tag{1.11}$$

may be regarded as a factor group of

$$R^l = S^m = T^n = RST = 1. \tag{1.111}$$

In terms of S and T, it takes the form

$$S^m = T^n = (ST)^l = (S^2T)^k = 1. \tag{1.12}$$

The group (1.111) is known* to be finite when

$$\frac{1}{l}+\frac{1}{m}+\frac{1}{n}>1,$$

and infinite otherwise. When $n=2$, we denote it by $[l, m]'$, since it is the rotation group of the regular polyhedron† $\{l, m\}$. Its factor group $(l, m\,|\,2, k)$ is the rotation group of the regular skew polyhedron‡ $\{l, m\,|\,k\}$.

The dihedral group $[2, m]'$, defined by $R^2=T^2=(RT)^m=1$, may be denoted more briefly by $[m]$, since it is the complete group of the regular m-gon $\{m\}$. Finally, $[m]'$ denotes the cycle group of order m, defined by $S^m=1$.

By interchanging R and S in (1.11), we get $(l, m\,|\,n, k)\sim(m, l\,|\,n, k)$. Again, writing R^{-1} for R, we have $(l, m\,|\,n, k)\sim(l, m\,|\,k, n)$. Hence

$$(l, m \mid n, k) \sim (m, l \mid n, k) \sim (m, l \mid k, n) \sim (l, m \mid k, n). \tag{1.13}$$

When naming special groups, we shall usually arrange the symbols so that $l\leq m$, $n\leq k$.

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* Threlfall [1], p. 28 ($r=3$).

† Todd [1], p. 214. When $1/l+1/m<1/2$, the polyhedron can be constructed in the "Minkowskian" space whose dimensions are two space-like and one time-like; see Coxeter [4], p. 24.

‡ Coxeter [7], p. 48.

When $l=2$, (1.11) takes the form

$$R^2 = S^n = (RS)^n = (RS)^k = 1.$$

Hence

(1.14) $(2, m \mid n, k)$ *collapses (partially or totally) if* $n \neq k$,

but

(1.141) $$(2, m \mid n, n) \sim [m, n]'$$

(finite when $1/m+1/n>1/2$). In particular,

(1.142) $$(2, 2 \mid n, n) \sim (2, n \mid 2, 2) \sim [n]$$

(where $[n]$ is the dihedral group of order $2n$).

When $l=3$, (1.11) takes the form

$$R^3 = S^m = (RS)^n = (R^{-1}S)^k = 1,$$

where, of course, R^{-1} may be replaced by R^2. Putting $S'=RS$, we derive

$$R^3 = (R^{-1}S')^m = S'^n = (RS')^k = 1.$$

Hence

(1.15) $$(3, m \mid n, k) \sim (3, n \mid k, m) \sim (3, k \mid m, n).$$

By (1.14) and (1.15),

(1.16) $(3, m \mid 2, k)$ *collapses if* $m \neq k$,

but

(1.161) $$(3, m \mid 2, m) \sim [3, m]'$$

(finite when $m<6$).

When $m=5$ and $n=2$, (1.12) takes the form

$$S^5 = T^2 = (ST)^l = (S^2T)^k = 1.$$

Putting $S'=S^2$, we derive

$$S'^5 = T^2 = (S'^3T)^l = (S'T)^k = 1.$$

Here, S'^3T may be replaced by S'^2T (conjugate to its inverse). Hence

(1.17) $$(l, 5 \mid 2, k) \sim (k, 5 \mid 2, l).$$

By (1.16) and (1.17),

(1.18) $(k, 5 \mid 2, 3)$ *collapses if* $k \neq 5$,

but

(1.181) $$(5, 5 \mid 2, 3) \sim [3, 5]'$$

(where $[3, 5]'$ is the icosahedral group).

Consider now the group $(l, m \mid 2, 2)$, defined by

$$R^l = S^m = (RS)^2 = (R^{-1}S)^2 = 1.$$

Let $U = SR$, $V = R^{-1}S$, and suppose that $m = 2q+1$. Then

$$S^{-1} = S^{m-1} = S^{2q} = (UV)^q,$$

and

$$R = S^{-1}U = (UV)^q U.$$

Since $U^2 = V^2 = 1$, it follows that $R^2 = 1$. Hence

(1.19) $(l, m \mid 2, 2)$ *collapses if* $l > 2$, m *odd.*

We have already seen (1.142) that $(2, m \mid 2, 2) \sim [m]$. The case when l and m are both even will be considered for general values of n and k.

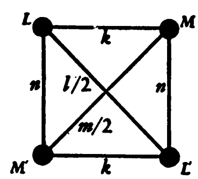
1.2. The criterion for finiteness, when l and m are even. The group $(l, m \mid n, k)$ clearly possesses an automorphism which replaces the generators R and S by their inverses. Let T denote an involutory operator which transforms the group according to this automorphism. By adjoining T, we derive the new group

(1.21) $$R^l = S^m = T^2 = (RT)^2 = (ST)^2 = (RS)^n = (R^{-1}S)^k = 1,$$

whose order is twice that of $(l, m \mid n, k)$.

Now suppose that l and m are even, and consider the *group generated by reflections**

(1.22) $$\begin{aligned} L^2 = L'^2 = M^2 = M'^2 = (LL')^{l/2} = (MM')^{m/2} \\ = (LM')^n = (ML')^n = (LM)^k = (L'M')^k = 1. \end{aligned}$$



This possesses an automorphism which interchanges L and L', M and M'. Let T denote an involutory operator which transforms the group according to this automorphism. We adjoin T by inserting the extra relations

* Coxeter [4]. In that paper, Lemmas 1 and 2 are unnecessary. In the proof of Lemma 3, after obtaining the invariant form $\sum c_i y_i^2$, we can introduce a euclidean metric by writing $x_i = c_i^{1/2} y_i$.

$$T^2 = 1, \qquad LT = TL' = R \text{ (say)},$$
$$MT = TM' = S \text{ (say)}.$$

The generators L, L', M, M' may now be eliminated by substituting

$$L = RT, \qquad L' = TR, \qquad M = ST, \qquad M' = TS;$$

and the augmented group takes the form

$$T^2 = (RT)^2 = (ST)^2 = R^l = S^m$$
$$= (RS)^n = (R^{-1}S)^k = 1,$$

which is the same as (1.21).

If l and m are even, $(l, m\,|\,n, k)$ has the same order as the above group generated by reflections, both being subgroups of index two in the same larger group.

This result enables us to assert that $(l, m\,|\,n, k)$ never collapses, as long as l and m are even and greater than 2 (see (1.14)*). Moreover, it provides a definite criterion for finiteness. For we know† that the group generated by reflections (1.22) is finite or infinite according as Δ is or is not positive, where

$$\Delta = \begin{vmatrix} 1 & -\cos 2\pi/l & -\cos \pi/k & -\cos \pi/n \\ -\cos 2\pi/l & 1 & -\cos \pi/n & -\cos \pi/k \\ -\cos \pi/k & -\cos \pi/n & 1 & -\cos 2\pi/m \\ -\cos \pi/n & -\cos \pi/k & -\cos 2\pi/m & 1 \end{vmatrix}$$

$$= \{(1 + \cos 2\pi/l)(1 + \cos 2\pi/m) - (\cos \pi/n - \cos \pi/k)^2\}$$
$$\cdot \{(1 - \cos 2\pi/l)(1 - \cos 2\pi/m) - (\cos \pi/n + \cos \pi/k)^2\}$$
$$= (2 \cos \pi/l \cos \pi/m + |\cos \pi/n - \cos \pi/k|)$$
$$\cdot (2 \cos \pi/l \cos \pi/m - |\cos \pi/n - \cos \pi/k|)$$
$$\cdot (2 \sin \pi/l \sin \pi/m + \cos \pi/n + \cos \pi/k)$$
$$\cdot (2 \sin \pi/l \sin \pi/m - \cos \pi/n - \cos \pi/k).$$

Of these four factors, the first three are essentially positive $(l, m > 2)$, the first and third obviously, and the second because l and m, being even, must be at least 4, so that

$$2 \cos \pi/l \cos \pi/m \geqq 1 \geqq |\cos \pi/n - \cos \pi/k|.$$

Thus the criterion for finiteness reduces to

$$(1.23) \qquad 2 \sin \pi/l \sin \pi/m > \cos \pi/n + \cos \pi/k.$$

* If $l = 2$, the g.g.r. has only three generators, since $L' = L$. It follows, even from the present point of view, that, to avoid collapse, we must have $n = k$.

† Coxeter [1], p. 137; [2], p. 602; [4], p. 24.

1.3. **Particular cases.** The finite groups, according to this criterion, are

$$(2i, 2j \mid 2, 2), (4, 4 \mid 2, k), (4, 6 \mid 2, 3), (4, 8 \mid 2, 3).$$

The orders of these are equal to the orders of the *groups generated by reflections**

$$[i, 2, j], \ [k, 2, k], \ [3, 3, 3], \ [3, 4, 3],$$

namely $4ij$, $4k^2$, 120, 1152.

The group $(2i, 2j \mid 2, 2)$, defined by

$$S^{2j} = T^2 = (ST)^{2i} = (S^2T)^2 = 1,$$

has an invariant cyclic subgroup of order j (generated by S^2), whose quotient group is the dihedral group of order $4i$. (It has also, of course, a cyclic subgroup of order i, whose quotient group is the dihedral group of order $4j$.) Returning to the definition in terms of R and S, we easily verify that $(2i, 2j \mid 2, 2)$ is generated by the permutations†

$$(1.31) \qquad \begin{cases} R = (a_1\, a_2 \cdots a_{2i})(b_1\, b_{2j})(b_2\, b_{2j-1}) \cdots (b_j\, b_{j+1}), \\ S = (b_1\, b_2 \cdots b_{2j})(a_1\, a_{2i})(a_2\, a_{2i-1}) \cdots (a_i\, a_{i+1}). \end{cases}$$

The group $(4, 4 \mid 2, k)$ is well known as one of the *groups of genus one.*‡ It is generated (when $k > 2$) by the permutations

$$(1.32) \qquad R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) \cdots, \qquad S = (1\ 2)(3\ 4\ 5\ 6) \cdots,$$

where the final cycles (of two or four numbers) end with $2k$.

The group $(4, 6 \mid 2, 3)$ is the symmetric group of degree five,§ generated by the permutations (1 4 2 5), (1 5)(2 3 4).

The group $(4, 8 \mid 2, 3)$ has a central of order two‖ whose quotient group is generated by the permutations (1 8)(2 7 3 6)(4 5), (1 6 3 7 4 5 2 8).

For all larger values of n and k, and for all larger even values of l and m, $(l, m \mid n, k)$ is infinite; for example, $(4, 4 \mid 3, 3)$, $(4, 6 \mid 2, 4)$, $(6, 6 \mid 2, 3)$, and $(4, 10 \mid 2, 3)$ are infinite.

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* Todd [1], p. 224, (4). Their graphical symbols (Coxeter [4], p. 21) being disconnected, the first two of these are direct products of dihedral groups, namely $[i] \times [j]$, $[k] \times [k]$.

† In this case

$$L = (a_1\, a_{2i-1})(a_2\, a_{2i-2}) \cdots (a_{i-1}\, a_{i+1}), \quad L' = (a_2\, a_{2i})(a_3\, a_{2i-1}) \cdots (a_i\, a_{i+2}),$$

M and M' are analogous, with b and j in place of a and i, and

$$T = (a_1\, a_{2i})(a_2\, a_{2i-1}) \cdots (a_i\, a_{i+1})(b_1\, b_{2j})(b_2\, b_{2j-1}) \cdots (b_j\, b_{j+1}).$$

‡ Edington [1], pp. 197–202. In this case $L = (2\ 4)(6\ 8) \cdots$, $M = (4\ 6)(8\ 10) \cdots$, $L' = (1\ 3)(5\ 7) \cdots$, $M' = (3\ 5)(7\ 9) \cdots$, and $T = (1\ 2)(3\ 4)(5\ 6) \cdots$.

§ In this case $L = (1\ 2)$, $M = (2\ 3)$, $M' = (3\ 4)$, $L' = (4\ 5)$, and $T = (1\ 5)(2\ 4)$. For the geometrical aspect, see Coxeter [7], p. 49.

‖ Since the polytope $\{3, 4, 3\}$ has central symmetry.

1.4. **The criterion when l and k are even and $n=2$.** We observe that the criterion (1.23) remains valid when $l=2$ and $n=k$; it then has wider scope, since m may be odd. Moreover, it agrees with the fact that $(3, m|2, m)$ is finite only when $m<6$. This suggests its possible significance in yet other cases. We shall see that it is applicable to the whole class of groups $(l, m|2, k)$ for which l and k are even, and greater than 2, while $m>3$ (but may be odd).

According to the criterion, $(l, m|2, k)$ should be finite when $l=k=4$ and $m=5$, but infinite for all greater even values of l and k, and for all greater values of m. Elsewhere* we saw that the group $(2p_1, m|2, 2p_2)$ is finite when $m=5$ and $p_1=p_2=2$, but infinite for all greater values of m, p_1, p_2. Since the argument is rather subtle, an outline of it is repeated here.

From the group generated by reflections

$$(1.41) \qquad R_i^2 = (R_iR_{i+1})^{l/2} = (R_iR_{i+2})^{k/2} = 1, \qquad R_{i+m} = R_i$$

(implying $l=k$ when $m=3$), we derive the group

$$S^m = R_0^2 = (R_0S^{-1}R_0S)^{l/2} = (R_0S^{-2}R_0S^2)^{k/2} = 1,$$

whose order is m times as great, by adjoining an operator S, of period m, which cyclically permutes the R's (so that $R_i=S^{-i}R_0S^i$). The augmented group and $(l, m|2, k)$ are of the same order, having a common subgroup of index two. The group (1.41), and so also $(l, m|2, k)$, is infinite when $m>5$, since R_iR_{i+3} is then of unspecified (and therefore infinite) period. It is known also to be infinite when $m=5$ and $l/2$ or $k/2>2$.

When $m=5$ and $l/2=k/2=2$, (1.41) is the abelian group of order 32 and type (1, 1, 1, 1, 1), generated (say) by the transpositions

$$R_i = (i\ i+5), \qquad i = 0, 1, 2, 3, 4.$$

Hence† $(4, 5|2, 4)$, of order $5\times 32=160$, is generated (in the form (1.12)) by the permutations

$$(1.42) \qquad S = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9), \qquad T = (1\ 9)(2\ 8)(3\ 7)(4\ 6).$$

(The "common subgroup" is generated by S and $(0\ 9\ 8\ 7\ 6)(5\ 4\ 3\ 2\ 1)$; T interchanges these.)

1.5. **Theorem** A. We summarize the results of §1.1, §1.2, and §1.4 in the following theorem:

THEOREM A. *The groups $(2, m|n, k)$ for $n\neq k$, $(3, m|2, k)$ for $m\neq k$, $(5, m|2, 3)$ for $m\neq 5$, and $(l, m|2, 2)$ for $l>2$ and m odd, all collapse. Apart*

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* Coxeter [6], p. 284, (4.9).

† Coxeter [6], p. 284.

from these cases, if l and m are even, or if l and k are even and* $n=2$, *the group* $(l, m \mid n, k)$ *is finite when*

$$2 \sin \pi/l \sin \pi/m > \cos \pi/n + \cos \pi/k,$$

and infinite otherwise.

The groups which this theorem shows to be finite are

$$(l, m \mid 2, 2)\ (l \text{ and } m \text{ even}),\quad (2, 2 \mid n, n),$$
$$(2, 4 \mid 3, 3),\quad (3, 4 \mid 2, 4),\quad (4, 4 \mid 2, k),$$
$$(4, 5 \mid 2, 4),\quad (4, 6 \mid 2, 3),\quad (4, 8 \mid 2, 3).$$

(See Table I, at the end of the paper.)

1.6. **Other groups that satisfy the criterion.** The remaining solutions of (1.23) are

$$(2, 3 \mid 4, 4),\quad (2, 3 \mid 5, 5),\quad (2, 5 \mid 3, 3),$$
$$(3, m \mid 3, 3)\ (m < 6),\qquad (3, 3 \mid 3, k),$$
$$(3, 3 \mid 4, 4),\quad (3, 4 \mid 3, 4),\quad (3, 5 \mid 2, 5),$$
$$(4, 5 \mid 2, 5),\quad (4, 7 \mid 2, 3),\quad (5, 5 \mid 2, 3).$$

By (1.141),

(1.61) $$(2, 3 \mid 4, 4) \sim [3, 4]' \sim G_{4!},$$

the octahedral group. Similarly, by (1.141), (1.161), and (1.181),

(1.62) $$(2, 5 \mid 3, 3) \sim (2, 3 \mid 5, 5) \sim (3, 5 \mid 2, 5) \sim (5, 5 \mid 2, 3) \sim G_{5!/2},$$

the icosahedral group.

The group $(3, 3 \mid 3, k)$ is another of the *groups of genus one.*† It is generated by the permutations

(1.63) $$(1\ 2\ 3)(4\ 5\ 6) \cdots (\cdot\cdot\ 3k),\qquad (3k\ 1\ 2)(3\ 4\ 5) \cdots (\cdot\cdot\ 3k-1),$$

and is of order $3k^2$. By (1.15),

(1.64) $$(3, k \mid 3, 3) \sim (3, 3 \mid 3, k).$$

Miller‡ proved that $(3, 3 \mid 4, 4)$ is Klein's 168-group, generated by permutations of the form

(1.65) $$R = (0\ 1\ \infty)(2\ 6\ 4),\qquad S = (0\ 2\ 3)(1\ 6\ 5),$$

* Perhaps $(5, m \mid 2, 3)$ should not have been mentioned, since its l and k are odd; but it seemed desirable to bring together *all* the known cases of collapse.

† Edington [1], p. 208.

‡ Miller [1], p. 364.

which are equivalent to the linear fractional substitutions

$$\left(\frac{1}{1-x}\right), \qquad (4x+2) \pmod 7.$$

By enumerating cosets* of the octahedral subgroup generated by RSR and S, we easily verify the order 168 and obtain the alternative representation

$$R = (1\ 2\ 3)(4\ 5\ 6), \qquad S = (2\ 3\ 4)(5\ 7\ 6).$$

The group $(4, 7 \mid 2, 3)$ is the same, since it can be derived from Burnside's†

$$S_2^2 = S_7^7 = (S_7S_2)^3 = (S_7^4S_2)^4 = 1$$

by putting $S_7 = S^2$, $S_2 = T$. Thus

(1.66) $$(3, 4 \mid 3, 4) \sim (3, 3 \mid 4, 4) \sim (4, 7 \mid 2, 3) \sim \mathfrak{P}_1(7).$$

Finally, by (1.17),

(1.67) $$(4, 5 \mid 2, 5) \sim (5, 5 \mid 2, 4) \sim G_{61/2},$$

$(5, 5 \mid 2, 4)$ having been used‡ as an example to illustrate the "Todd-Coxeter method" for enumerating cosets.

These results show that $(l, m \mid n, k)$ is finite (or collapses) whenever (1.23) is satisfied; that is, whenever

$$2 \sin \pi/l \sin \pi/m > \cos \pi/n + \cos \pi/k.$$

Hence we obtain the following theorem:

THEOREM B. *For all infinite groups* $(l, m \mid n, k)$,

$$2 \sin \pi/l \sin \pi/m \leqq \cos \pi/n + \cos \pi/k.$$

1.7. **Finite groups which violate the criterion.** The converse is false, since $(5, 5 \mid 2, 4)$, which violates (1.23), is the same group as $(4, 5 \mid 2, 5)$, which satisfies it. So too, $(3, m \mid 3, 3)$ violates (1.23) when $m \geqq 6$, although $(3, 3 \mid 3, k)$ satisfies it universally. We proceed to describe six further cases, namely:

(1.71) $(3, 3 \mid 4, 5) \sim (3, 4 \mid 3, 5) \sim (3, 5 \mid 3, 4)$, of order 1080,

(1.72) $(6, 7 \mid 2, 3) \sim (7, 7 \mid 2, 3) \sim \mathfrak{P}_1(13)$,§

(1.73) $(4, 9 \mid 2, 3) \sim \mathfrak{P}_1(17)$.

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* Todd and Coxeter [1].

† Burnside [1], p. 422.

‡ Todd and Coxeter [1], p. 31, (5).

§ The fact that $(6, 7 \mid 2, 3)$ and $(7, 7 \mid 2, 3)$ are finite, while $(6, 6 \mid 2, 3)$ is infinite, shows that any attempt to "improve" the criterion would be futile.

The group (3, 3 | 4, 5) shows the efficiency of the Todd-Coxeter method so clearly that I shall perhaps be forgiven for writing out the work in full. Taking the icosahedral subgroup generated by RSR and S (and using rows instead of columns), we proceed with the enumeration of cosets as follows:

R	R	R	
1	2	3	1
4	5	6	4
7	8	9	7
10	11	12	10
13	14	15	13
16	17	18	16

S	S	S	
1	1	1	1
2	3	4	2
5	7	10	5
6	11	8	6
9	16	17	9
12	13	14	12
15	15	15	15
18	18	18	18

R	S	R	S	R	S	R	S	
1	2	3	1	1	2	3	1	1
2	3	4	5	7	8	6	4	2
11	12	13	14	12	10	5	6	11
16	17	9	7	10	11	8	9	16
15	13	14	15	15	13	14	15	15
18	16	17	18	18	16	17	18	18

R^{-1}	S	R^{-1}	S	R^{-1}	S	R^{-1}	S	R^{-1}	S	
1	3	4	6	11	10	5	4	2	1	1
3	2	3	2	3	2	3	2	3	2	3
7	9	16	18	18	17	9	8	6	5	7
10	12	13	15	15	14	12	11	8	7	10
14	13	14	13	14	13	14	13	14	13	14
17	16	17	16	17	16	17	16	17	16	17

This verifies the order, $18 \times 60 = 1080$, which was first obtained by Miller,* and gives us the permutations

$$(1.74) \quad \begin{cases} R = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15)(16\ 17\ 18), \\ S = (2\ 3\ 4)(5\ 7\ 10)(6\ 11\ 8)(9\ 16\ 17)(12\ 13\ 14). \end{cases}$$

(To see how much labor has been saved, compare pages 365–368 of Miller's paper.) The permutation

$$(1.741) \quad \begin{aligned} Z &= (1\ 15\ 18)(2\ 13\ 16)(3\ 14\ 17)(4\ 12\ 9)(5\ 10\ 7)(6\ 11\ 8) \\ &= (R^{-1}S^{-1}RS)^5 \end{aligned}$$

generates the central, of order three, whose quotient group† is the alternating group $G_{61/2}$, generated by (1 2 3)(4 5 6) and (2 3 4).

The groups (6, 7 | 2, 3) and (7, 7 | 2, 3) are equivalent to definitions for $\mathfrak{P}_1(13)$ which were given by Brahana and Sinkov, respectively.‡ Identifying

* Miller [1], p. 368.

† (3, 3 | 4, 5; 5), in the notation of the footnote to p. 74 above.

‡ Brahana [2], p. 354; Sinkov [1], p. 239; Coxeter [7], p. 56.

our T with theirs, namely

$$T = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7), \tag{1.75}$$

we find that the other generator of $(6, 7 \mid 2, 3)$ or $(7, 7 \mid 2, 3)$ (namely, S) is, in Sinkov's notation,* $(S_{11}T)^3$ or $(S_{13}T)^3$, respectively. Thus $(6, 7 \mid 2, 3)$ is generated by T and

$$S = (0\ 3\ 10\ 8\ 2\ 5\ 9)(1\ 6\ 7\ \infty\ 11\ 12\ 4), \tag{1.751}$$

while $(7, 7 \mid 2, 3)$ is generated by the same T and

$$S = (0\ 12\ 9\ 1\ 11\ 10\ 5)(2\ 3\ 7\ 8\ 6\ \infty\ 4). \tag{1.752}$$

In the case of $(4, 9 \mid 2, 3)$, by enumerating the 272 cosets of the cyclic subgroup generated by S, I found the order to be 2448. This irresistibly suggested (1.73), which Sinkov showed to be true.† He cited two linear fractional substitutions (mod 17) which lead to the permutations

$$S = (0\ 6\ 10\ 14\ 3\ 4\ 15\ 5\ 16)(1\ 13\ \infty\ 7\ 2\ 12\ 11\ 9\ 8),$$
$$T = (1\ 16)(2\ 15)(3\ 14)(4\ 13)(5\ 12)(6\ 11)(7\ 10)(8\ 9).$$

Although there are, as we have seen, infinitely many finite groups which violate the criterion (1.23), it still seems improbable that all the unknown groups $(l, m \mid n, k)$ will turn out to be finite. On the contrary, my conjecture is that they are all infinite. We shall see later (2.67) that $(3, 3 \mid 4, 6)$ is certainly infinite.

Chapter II. $(l, m, n; q)$

2.1. **Various forms of the abstract definition.** The group $(l, m, n; q)$, defined by

$$R^l = S^m = (RS)^n = (R^{-1}S^{-1}RS)^q = 1, \tag{2.11}$$

involves the numbers l, m, n symmetrically, since its definition is equivalent to

$$R^l = S^m = T^n = RST = (TSR)^q = 1. \tag{2.111}$$

When $n=2$, so that $RS=S^{-1}R^{-1}$, we have‡ $RS^{-1}R^{-1}S=R^2S^2$. Hence $(2, l, m; q)$ is defined by

$$R^l = S^m = (RS)^2 = (R^2S^2)^q = 1. \tag{2.12}$$

When $m=3$ and $n=2$, so that $TR^{-1}T=RTR$, we have $TR^{-1}TR=RTR^2$,

* Sinkov [3], p. 71. I have replaced Sinkov's symbols 13 and 14 by 0 and ∞, respectively.
† Coxeter [7], p. 56.
‡ Edington [1], p. 195.

which is conjugate to R^3T. Thus the definition

$$R^l = T^2 = (TR)^3 = (TR^{-1}TR)^q = 1$$

for $(2, 3, l; q)$ is equivalent to

$$R^l = T^2 = (RT)^3 = (R^3T)^q = 1. \tag{2.13}$$

2.2. **The abelian case** $(q=1)$. We proceed to prove that $(l, m, n; 1)$ *collapses unless it is of the form* $(bcd, cad, abd; 1)$, *where* a, b, c *are co-prime in pairs*, and that it is then the direct product of cyclic groups of orders $abcd$ and d.

Consider the abelian group $(l, m, n; 1)$ in the form

$$R^l = S^m = (RS)^n = 1, \qquad RS = SR. \tag{2.21}$$

Let N denote the least common multiple of l and m. Then

$$(RS)^N = R^NS^N = 1;$$

so N must be a multiple of n (or the group would collapse). Thus, on account of the symmetry between l, m, n, each of these three members must divide the least common multiple of the other two.

Let $mn/l=\lambda$, $nl/m=\mu$, $lm/n=\nu$. Then $l^2=\mu\nu$, $m^2=\nu\lambda$, $n^2=\lambda\mu$; so we may write $\lambda=a^2d$, $\mu=b^2d$, $\nu=c^2d$, where a, b, c, d are positive integers. Thus

$$l = bcd, \qquad m = cad, \qquad n = abd. \tag{2.22}$$

By absorbing into d any common factor of a, b, c, we may suppose* that $(a, b, c)=1$.

Further, since $N=abcd/(a, b)$ must be a multiple of $n=abd$, c must be a multiple of (a, b). Hence $(a, b, c)=(a, b)$ and

$$(b, c) = (c, a) = (a, b) = 1. \tag{2.23}$$

We are thus led to consider the group $R^{bcd}=S^{cad}=(RS)^{abd}=1$, $RS=SR$, where a, b, c are co-prime in pairs. Accordingly, we may choose (positive or negative) integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$, such that

$$\alpha b - \beta a = 1, \qquad \gamma a - \alpha' c = 1, \qquad \beta' c - \gamma' b = 1.$$

Then

$$(R^{\beta a}S^{\alpha b})^{\alpha bcd} = S^{\alpha^2b^2cd} = S^{(\beta a+1)^2cd} = S^{cd},$$
$$(R^{\beta a}S^{\alpha b})^{\beta cad} = R^{\beta^2a^2cd} = R^{(\alpha b-1)^2cd} = R^{cd},$$
$$(R^{\beta a}S^{\alpha b})^{\gamma abd} = R^{(\beta a-\alpha b)\gamma abd} = R^{-(\alpha' c+1)bd} = R^{-bd},$$
$$(R^{\beta a}S^{\alpha b})^{\gamma' abd} = S^{(\alpha b-\beta a)\gamma' abd} = S^{(\beta' c-1)ad} = S^{-ad},$$

* Following the usual notation, we let $(a, b, \cdots)$ denote the greatest common factor of $a, b, \cdots$.

and therefore

$$R^d = R^{(\beta' c-\gamma' b)d} = (R^{\beta a}S^{\alpha b})^{(\beta'\beta c+\gamma'\gamma b)ad},$$
$$S^d = S^{(\gamma a-\alpha' c)d} = (R^{\beta a}S^{\alpha b})^{-(\gamma\gamma' a+\alpha'\alpha c)bd},$$
$$(RS)^d = R^dS^d = (R^{\beta a}S^{\alpha b})^{(\beta'\beta a-\alpha'\alpha b)cd}.$$

Since $S=R^{\beta a}S^{\alpha b}(RS)^{-\beta a}$, it follows that every operator of the group is expressible in the form

$$(R^{\beta a}S^{\alpha b})^x(RS)^y, \qquad 0 \leqq y < d,$$

and that the group is the direct product of cyclic groups, of orders *abcd* and *d*, generated by

$$R^{\beta a}S^{\alpha b}, \qquad (R^{\beta a}S^{\alpha b})^{(\alpha'\alpha b-\beta'\beta a)c}RS.$$

Thus, under the condition (2.23),

$$(2.24) \qquad (bcd,\ cad,\ abd;\ 1) \sim [abcd]' \times [d]',$$

the product of two cyclic groups.

2.3. **The polyhedral groups and** (2, 3, 7; q). When $l=m=2$, the last relation of (2.11) is $(RS)^{2q}=1$; so (2, 2, n; q) collapses unless either $n=2q$, or $n=q$ and n is odd. In the former case we have the dihedral group

$$(2.31) \qquad (2,\ 2,\ 2q;\ q) \sim [2q],$$

and in the latter,

$$(2.311) \qquad (2,\ 2,\ q;\ q) \sim [q], \qquad q \text{ odd}.$$

The definition (2.13) enables us easily to evaluate q for the polyhedral groups $[3, l]'$, ($l<6$), regarded as (2, 3, l; q). For the *tetrahedral group* ($l=3$), $R^3T=T$; so $q=2$. For the *octahedral group* ($l=4$), $R^3T=(TR)^{-1}$; so $q=3$. For the *icosahedral group* ($l=5$), $R^3T=(TR^2)^{-1}$; so

$$(2.32) \qquad (2,\ 3,\ 5;\ q) \sim (3,\ 5 \mid 2,\ q),$$

whence, by (1.16), $q=5$. Thus

$$(2.33) \qquad (2,\ 3,\ 3;\ 2) \sim [3,\ 3]' \sim G_{4!/2},$$

$$(2.34) \qquad (2,\ 3,\ 4;\ 3) \sim [3,\ 4]' \sim G_{4!},$$

$$(2.35) \qquad (2,\ 3,\ 5;\ 5) \sim [3,\ 5]' \sim G_{5!/2}.$$

In each case, any other value of q would cause collapse.

In the form (2, 3, 5; q)$\sim$(q, 5 | 2, 3) (see (1.17)), (2.32) remains true when the fives are replaced by sevens, although the obvious generalization fails. For, writing S^2 for R in (2.13) with $l=7$, we obtain

$$S^7 = T^2 = (S^2T)^3 = (ST)^q = 1.$$

Thus

$$(2.36) \qquad (2, 3, 7; q) \sim (q, 7 \mid 2, 3).$$

This provides an elegant proof for Brahana's result* that $(2, 3, 7; q)$ collapses when $q=2$, 3, or 5. (We use (1.14), (1.16), and (1.18), respectively.) His definition†

$$Q^q = R^7 = (QR^3)^2 = (QR^2)^3 = 1$$

is equivalent to (2.13) ($l=7$) with $T=QR^3$.

By (2.36) and (1.66),

$$(2.37) \qquad (2, 3, 7; 4) \sim \mathfrak{P}_1(7).$$

In the form

$$A_1^7 = A_2^3 = (A_1A_2)^2 = (A_2A_1^5)^4 = 1$$

(see (2.12)), this definition was given by Dyck.‡ Thus (2.36) (with $q=4$) relates Dyck's definition to Burnside's.

By (2.36) and (1.72), we have

$$(2.38) \qquad (2, 3, 7; 6) \sim (2, 3, 7; 7) \sim \mathfrak{P}_1(13).$$

These definitions are due to Brahana and Sinkov, respectively.§ By (1.75) and (1.751), $(2, 3, 7; 6)$ is generated by

$$(2.39) \qquad \begin{cases} R = (0\ 10\ 2\ 9\ 3\ 8\ 5)(1\ 7\ 11\ 4\ 6\ \infty\ 12), \\ T = (1\ 12)(2\ 11)(3\ 10)(4\ 9)(5\ 8)(6\ 7); \end{cases}$$

and by (1.752), $(2, 3, 7; 7)$ is generated by the same T and

$$(2.391) \qquad R = (0\ 9\ 11\ 5\ 12\ 1\ 10)(2\ 7\ 6\ 4\ 3\ 8\ \infty).$$

Sinkov has shown‖ that $(2, 3, 7; 8)$ has a factor group of order 10752. When written as a factor group of $(8, 7 \mid 2, 3)$, this takes the form

$$R^8 = S^7 = (RS)^2 = (R^{-1}S)^3 = (R^2S^4)^6 = 1.$$

2.4. **A lemma on automorphisms.** We shall often have occasion to use the following general principle, which the reader will have no difficulty in proving.

* Brahana [2], pp. 350, 352.

† Ibid., p. 349; quoted by Sinkov [3], p. 68.

‡ Dyck [1], p. 41; Burnside [1], p. 422.

§ Brahana [2], p. 354; Sinkov [1], p. 239.

‖ Sinkov [5].

If a group G is augmented by the adjunction of an operator T, of period n, which transforms G according to an *inner* automorphism of the same period, and if the order of the central of G is prime to n, then the augmented group is the direct product $G\times\{T\}$.

The necessity* for mentioning the central is illustrated by the following example, due to R. Brauer. Let G be the trigonal dicyclic group† (of order 12), $B^3=C^4=1$, $C^{-1}BC=B^{-1}$, with generators $B=(1\ 2\ 3)$, $C=(2\ 3)(4\ 5\ 6\ 7)$. Then the operator $T=(2\ 3)$ transforms G in the same manner as C. But $\{B, C, T\}$, being generated by (1 2 3), (2 3), and (4 5 6 7), is the direct product of the symmetric group of order six and the cyclic group of order four (*not* the direct product of the dicyclic group and the group of order two).

2.5. **The derivation of** $(2, m, m; q)$ **and** $(2, m, 2n; k)$ **from** $(4, m\,|\,2, 2q)$ **and** $(m, m\,|\,n, k)$. The group $R^m=S^m=(RS)^n=(R^2S^2)^q=(R^{-1}S)^k=1$ has an automorphism which interchanges the generators. We adjoin an involutory operator T, which transforms the group according to this automorphism. Substituting TST for R, we obtain the augmented group in the form‡ $S^m=T^2=(ST)^{2n}=(S^2T)^{2q}=(S^{-1}TST)^k=1$. This result provides two useful lemmas, the first by ignoring k and putting $n=2$, the second by ignoring q:

(2.51) $(2, m, m; q)$ *is a subgroup of index two in* $(4, m\,|\,2, 2q)$.

(2.52) $(2, m, 2n; k)$ *contains* $(m, m\,|\,n, k)$ *as a subgroup of index two.*

Since $(m, m\,|\,n, k)\sim(m, m\,|\,k, n)$, it follows that the two groups $(2, m, 2n; k)$, $(2, m, 2k; n)$ have the same order. Since $(m, m\,|\,k, n)$ is derived from $(m, m\,|\,n, k)$ by writing R^{-1} for R, we may say that $(2, m, 2k; n)$ is derived from $(m, m\,|\,n, k)$ by adjoining an involutory operator T', which transforms S into R^{-1} (and R into S^{-1}).

In particular,§ since $(4, 4\,|\,2, k)$ is of order $4k^2$, both $(2, 4, 4; k)$ and $(2, 4, 2k; 2)$ are of order $8k^2$. The order of $(2, 4, 2k; 2)$ can alternatively be deduced from that of its subgroup $(2k, 2k\,|\,2, 2)$ which was described in §1.3. Again,|| $(2, 3, 6; k)$ and $(2, 3, 2k; 3)$ are of order $6k^2$, since both contain $(3, 3\,|\,3, k)$ as a subgroup of index two.

By (2.51), $(2, 4, 4; k)$ is a subgroup of index two in $(4, 4\,|\,2, 2k)$, and $(2, 5, 5; 2)$ in $(4, 5\,|\,2, 4)$. By (1.42), it follows that¶ $(2, 5, 5; 2)$, *of order eighty, is generated by*

* Pointed out by a referee.

† In terms of BC^2 and C, this group has the simpler definition $A^3=C^2=(AC)^2$.

‡ That is, $(2n, m\,|\,2, 2q; k)$.

§ Edington [1], p. 197; Sinkov [2], p. 79.

|| Edington [1], p. 207; Sinkov [2], p. 82.

¶ Coxeter [6], p. 284.

(2.53) $S = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9)$, $TST = (0\ 9\ 8\ 7\ 6)(5\ 4\ 3\ 2\ 1)$.

A number of special applications of (2.52) may conveniently be arranged as columns of a table:

$(m, m \mid n, k)$	$(5, 5 \mid 2, 3) \sim G_{5!/2}$	$(3, 3 \mid 4, 4) \sim \mathfrak{P}_1(7)$
R S	(2 1 3 4 5) (1 2 3 4 5)	(0 1 ∞)(2 6 4) (0 2 3)(1 6 5)
Automorphism (R, S)	(1 2) (outer)	(1 2)(3 ∞)(4 5) (outer)
Automorphism (R^{-1}, S)	(3 5) (outer)	
$(2, m, 2n; k)$ $(2, m, 2k; n)$	$(2, 4, 5; 3) \sim G_{5!}$* $(2, 5, 6; 2) \sim G_{5!}$	$(2, 3, 8; 4) \sim \bar{\mathfrak{P}}_1(7)$

$(m, m \mid n, k)$	$(5, 5 \mid 2, 4) \sim G_{6!/2}$	$(7, 7 \mid 2, 3) \sim \mathfrak{P}_1(13)$
R S	(1 6 5 4 3)† (1 2 3 4 5)	(0 5 7 8 10 2 1)(3 11 4 12 9 ∞ 6) (5 0 12 9 1 11 10)(4 2 3 7 8 6 ∞)
Automorphism (R, S)	(2 6)(3 5) (inner)	(0 5)(1 10)(2 11)(3 4)(6 ∞)(7 12)(8 9) (outer)
Automorphism (R^{-1}, S)	(degree greater than 6) (outer)	(0 ∞)(1 4)(3 10)(5 6)(7 8)(9 12) (inner)‡
$(2, m, 2n; k)$ $(2, m, 2k; n)$	$(2, 4, 5; 4) \sim G_{6!/2} \times G_2$ (2, 5, 8; 2)(see below)	$(2, 4, 7; 3) \sim \bar{\mathfrak{P}}_1(13)$ $(2, 6, 7; 2) \sim \mathfrak{P}_1(13) \times G_2$

* Burnside [1], p. 422.

† Todd and Coxeter [1], p. 31.

‡ Since $S^2R^{-2} = (0\ 4\ \infty\ 8\ 12\ 10\ 2)(1\ 7\ 9\ 6\ 3\ 5\ 11)$, the relations $S^7 = T'^2 = (ST')^6 = (S^{-1}T'ST')^2 = 1$ imply $(S^2T')^{14} = 1$. Hence $(6, 7 \mid 2, 7; 2) \sim \mathfrak{P}_1(13)$.

In each of these cases, $(m, m \mid n, k)$ is a simple group; so the order of its central is 1. The interchange of R and S (or of R^{-1} and S) is usually recognizable as an inner or outer automorphism according as it is an even or odd permutation.

In describing (2, 3, 8; 4) and (2, 4, 7; 3), we appeal to the theorem (of Schreier and van der Waerden) that the group of isomorphisms of $\mathfrak{P}_1(p)$ (p prime) is $\tilde{\mathfrak{P}}_1(p)$. It is the same theorem which enables us to assert that the even permutation $(0\ \infty)(1\ 4)(3\ 10)(5\ 6)(7\ 8)(9\ 12)$ transforms $\mathfrak{P}_1(13)$ according to an inner automorphism.

In the case of $(5, 5 \mid 2, 4)$, since R^{-1} and S are not interchanged by any permutation of degree six, T' transforms $G_{6!/2}$ according to an outer automorphism of $G_{6!}$. In other words, (2, 5, 8; 2) (of order 720) is *the nonsymmetric subgroup of index two in the group of isomorphisms of* $G_{6!}$. It is easily verified that the following permutations of degree ten satisfy the defining relations for $(5, 5 \mid 2, 4)$ and so generate $G_{6!/2}$:

$$R = (4\ 7\ 8\ 3\ 0)(9\ 6\ 1\ 2\ 5), \qquad S = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9). \tag{2.54}$$

In this representation, R^{-1} and S are interchanged by

$$T' = (0\ 5)(1\ 2)(3\ 6)(4\ 9)(7\ 8).$$

Hence (2, 5, 8; 2) is generated by the permutations S and T'.

Finally, let us apply (2.52) to Miller's group $(3, 3 \mid 4, 5)$, of order 1080, so as to derive (2, 3, 8; 5) and (2, 3, 10; 4), of order 2160. Since, in (1.74), the permutation R involves six cycles, while S involves only five, the transformations effected by T and T' are outer automorphisms of $(3, 3 \mid 4, 5)$. But the combined operator TT', which transforms R and S into their own inverses, behaves like the permutation

$$P = (2\ 3)(5\ 6)(7\ 8)(10\ 11)(13\ 14)(16\ 17). \tag{2.55}$$

The generators

$$R_1 = (1\ 2\ 3)(4\ 5\ 6), \qquad S_1 = (2\ 3\ 4),$$

of the central quotient group of $(3, 3 \mid 4, 5)$, are transformed into their own inverses by

$$(2\ 3)(5\ 6) = (R_1S_1)^2(S_1R_1)^2(R_1S_1)^2(S_1R_1)^2(R_1S_1)^2.*$$

It is easily verified that

$$(RS)^2(SR)^2(RS)^2(SR)^2(RS)^2 = P.$$

Hence the transformation effected by TT' is an inner automorphism of $(3, 3 \mid 4, 5)$, and

$$(2, 3, 8; 5) \sim (2, 3, 10; 4). \tag{2.56}$$

* For this expression I am indebted to Dr. Sinkov.

By (2.13), $(2, 3, l; q)$ is the group of the *skew polyhedron** $\{3, l\,|, q\}$. Hence the two polyhedra $\{3, 8\,|, 5\}$ and $\{3, 10\,|, 4\}$ have the same group, as was implied in Table II† at the end of *Regular skew polyhedra in three and four dimensions, and their topological analogues.*

We have already remarked (1.741) that $(3, 3\,|\,4, 5)$ has a central, of order three, generated by‡ $Z=(R^{-1}S^{-1}RS)^5$. Since T (and likewise T') transforms this operator into its inverse, the group of order three generated by Z, regarded as a subgroup of $(2, 3, 8; 5)$ or $(2, 3, 10; 4)$, is no longer central although of course it is still invariant. Its quotient group, of order 720, is derived from $G_{6!/2}$ by adjoining an operator which transforms the latter according to an outer automorphism of $G_{6!}$, that is,

$$(2.57) \qquad (2, 3, 8; 5)/G_3 \sim (2, 5, 8; 2).$$

In fact, the generators of $G_{6!/2}$, as permutations of degree ten, take the form of

$$R_1 = (0\ 2\ 7)(1\ 8\ 4)(5\ 6\ 9), \qquad S_1 = (9\ 7\ 2)(8\ 1\ 5)(4\ 3\ 0).$$

These are interchanged by

$$T_1 = (0\ 9)(1\ 8)(2\ 7)(3\ 6)(4\ 5).$$

It is easily verified that S_1 and T_1 satisfy the relations

$$S_1^3 = T_1^2 = (S_1T_1)^8 = (S_1^{-1}T_1S_1T_1)^5 = 1.$$

2.6. **The criterion for finiteness.** By applying (2.51) and (2.52) to the general groups considered in Theorem A, we deduce the following theorem:

THEOREM C. *If $q>1$ and $1/m+1/n\leqq 1/2$, and if m and n are either even or equal (or both), the group $(2, m, n; q)$ is finite when*

$$(2.61) \qquad \cos 2\pi/m + \cos 2\pi/n + \cos \pi/q < 1,$$

and infinite otherwise.

For example, $(2, 4, 6; 3)$, $(2, 6, 6; 2)$, and $(2, 5, 5; 3)$ are infinite. The condition $1/m+1/n\leqq 1/2$ is inserted in order to exclude the polyhedral groups of §2.3, for which q is a function§ of m, n.

The groups $(2, 4, 2k; 2)$, $(2, 4, 4; q)$, and $(2, 5, 5; 2)$ are the only finite groups which satisfy all the conditions of Theorem C. If we allow m, n to be

* Coxeter [7], p. 59.

† Ibid., p. 61.

‡ Cf. Miller [1], p. 365, where Z is expressed in the form $S\{(S^{-1}R)^2(SR^{-1})^2\}^2S^{-1}$.

§ See (4.71), below.

unequal and (one or both) odd, we find that the criterion (2.61) admits the groups

$$(2, 3, 2k; 3), \quad (2, 3, 6; q), \quad (2, 3, 7; 4),$$
$$(2, 3, 7; 5) \text{ (collapsing)}, \quad (2, 3, 7; 6),$$
$$(2, 3, 8; 4), \quad (2, 4, 5; 3), \quad (2, 5, 6; 2),$$

which we have already discussed, and admits also the following further possibilities:

$$(2, 3, n; 2), \quad (2, 3, 9; 4), \quad (2, 5, 7; 2),$$
$$(2, 3, n; 3) \ (n \text{ odd}), \quad (2\ 4, n; 2) \ (n \text{ odd}).$$

Sinkov has proved* that $(2, 3, n; 2)$ for $n \neq 3, 6$, $(2, 4, n; 2)$ for n odd, and $(2, 3, n; 3)$ for n odd, all collapse. Since $(2, 4, n; 2)$ contains $(n, n \mid 2, 2)$ as a subgroup of index two, the collapse of $(2, 4, n; 2)$ (n odd) can alternatively be deduced from (1.19). For the collapse of (2, 3, 9; 4) and (2, 5, 7; 2), see the Appendix; in both cases the enumeration of cosets by the Todd-Coxeter method soon breaks down.

These results show that $(2, m, n; q)$ is finite (or collapses) whenever (2.61) is satisfied. Hence we have the following theorem:

THEOREM D. *For all infinite groups* $(2, m, n; q)$,

$$\cos 2\pi/m + \cos 2\pi/n + \cos \pi/q \geqq 1. \tag{2.62}$$

We saw, in §2.2 and §2.4, that the groups

$$(2,3,7;7), (2,3,8;5) \sim (2,3,10;4), (2,4,5;4), (2,4,7;3), (2,5,8;2), (2,6,7;2)$$

are finite, although they satisfy (2.62) (see §1.7). There are at least two further cases of this kind. By enumerating the 380 cosets of the subgroup of order nine generated by S, I found the order of (2, 5, 9; 2) to be 3420. Since the defining relations

$$S^9 = T^2 = (ST)^5 = (S^{-1}TST)^2 = 1$$

are satisfied by the linear fractional substitutions

$$S = (-3x), \qquad T = \left(\frac{x-7}{x-1}\right) \pmod{19},$$

we can therefore assert that

$$(2, 5, 9; 2) \sim \mathfrak{P}_1(19). \tag{2.63}$$

* Sinkov [2], pp. 75, 79, 82.

As permutations, we have

$$(2.64)\quad \begin{cases} S = (1\ 16\ 9\ 11\ 5\ 4\ 7\ 17\ 6)(2\ 13\ 18\ 3\ 10\ 8\ 14\ 15\ 12), \\ T = (0\ 7)(1\ \infty)(2\ 14)(3\ 17)(4\ 18)(5\ 9)(6\ 15)(8\ 11)(10\ 13)(12\ 16). \end{cases}$$

Again, by enumerating* the 552 cosets of the subgroup of order eleven generated by S, I found the order of $(2, 3, 11; 4)$ to be 6072. Since the defining relations

$$S^{11} = T^2 = (ST)^3 = (S^{-1}TST)^4 = 1$$

are satisfied by the linear fractional substitutions†

$$S = (9x), \qquad T = \left(\frac{x-3}{x-1}\right) \pmod{23},$$

we can therefore assert that

$$(2.65)\qquad (2, 3, 11; 4) \sim \mathfrak{P}_1(23).$$

As permutations, we have

$$(2.66)\begin{cases} S = (1\ 9\ 12\ 16\ 6\ 8\ 3\ 4\ 13\ 2\ 18)(5\ 22\ 14\ 11\ 7\ 17\ 15\ 20\ 19\ 10\ 21), \\ T = (0\ 3)(1\ \infty)(2\ 22)(4\ 8)(5\ 12)(6\ 19)(7\ 16)(9\ 18)(10\ 11)(13\ 20)(14\ 15)(17\ 21). \end{cases}$$

This result adds one to the list of *finite polyhedra* $\{l,\ m|,\ q\}$,‡ namely $\{3, 11|, 4\}$, with 2024 (triangular) faces, 3036 edges, 552 vertices, genus 231, and group $LF(2, 23)$, of order 6072.

On the other hand, Theorem C does not cover *all* the infinite groups $(2, m, n; q)$. For, Brahana§ has shown that $(2, 3, 8; 6)$ is infinite, whence, by (2.52),

$$(2.67)\qquad (3, 3\,|\,4, 6),\quad (2, 3, 12; 4)\quad \textit{are infinite}.$$

Fig. 4 (on p. 121) shows a graphical enumeration of groups $(2, n, p; 2)$. Known finite groups are represented by o's, known infinite groups by dots, and known cases of collapse by crosses.

2.7. **The derivation of** $(n, n, n; q)$, $(3, 3, 3; k)$, $(3, 3, 4; 3)$ **from** $(3, 3\,|\,n, 3q)$, $(3, 3\,|\,3, k)$, $(3, 3\,|\,4, 4)$. The group

$$R^n = S^n = T^n = RST = (TSR)^q = (R^{-1}S)^k = (S^{-1}T)^k = (T^{-1}R)^k = 1$$

---

* After some practice, such enumeration proceeds at the rate of about sixty cosets per hour.

† Cf. the substitutions $S=(4x)$, $T=[(x-5)/(x-1)]$ (mod 23), which satisfy $S^{11}=T^2=(ST)^4=(S^2T)^3=1$. However, the order of $(4, 11\,|\,2, 3)$ is certainly greater than 6072; so $\mathfrak{P}_1(23)$ is a proper factor group.

‡ Coxeter [7], p. 61, Table II.

§ Brahana [4], p. 901.

has an automorphism which cyclically permutes the three generators. We adjoin an operator Q^{-1}, of period three, which transforms the group according to this automorphism. Since $T=QSQ^{-1}$ and $R=Q^{-1}SQ$, the augmented group is defined by*

$$Q^3 = S^n = (QS)^3 = (Q^{-1}S)^{3q} = (S^{-1}Q^{-1}SQ)^k = 1.$$

By ignoring first k and then q, we deduce two lemmas analogous to (2.51) and (2.52):

(2.71) *$(n, n, n; q)$ is an invariant subgroup of index three in $(3, 3 \mid n, 3q)$.*

(2.72) *$(3, 3, n; k)$ contains*

$$R^n = S^n = T^n = RST = (R^{-1}S)^k = (S^{-1}T)^k = (T^{-1}R)^k = 1$$

as an invariant subgroup of index three.

This last group reduces to something familiar in two special cases. When $n=3$, since

$$S^{-1}T = S^2T = SR^{-1}, \qquad RT^{-1} = R^{-2}T^{-1} = R^{-1}S,$$

we are left with

$$R^3 = S^3 = (RS)^3 = (R^{-1}S)^k = 1.$$

Since $(3, 3 \mid 3, k)$ is of order $3k^2$, it follows that $(3, 3, 3; k)$ is of order $9k^2$.

Secondly, when $n=4$ and $k=3$, since

$$\begin{cases} SR^{-1} = S^2T, \\ R^{-1}T = ST^2 = ST^{-1}\cdot T^{-1} = (TS^{-1})^2T^{-1} = TS\cdot S^2T\cdot S^{-1}T^{-1}, \\ (S^2T)^2R\cdot S^2T = S^2TS^{-1}T = S^2\cdot ST^{-1}S = S^{-1}T^{-1}S, \end{cases}$$

we are left with

$$S^4 = T^4 = (S^{-1}T)^3 = (S^2T)^3 = 1$$

(compare (1.12) with S^{-1} for S). Since $(3, 4 \mid 3, 4) \sim \mathfrak{P}_1(7)$, it follows that $(3, 3, 4; 3)$ is of order 504. We easily verify that $\mathfrak{P}_1(7)$, in the form

$$R^4 = S^4 = T^4 = RST = (R^{-1}S)^3 = (S^{-1}T)^3 = (T^{-1}R)^3 = 1,$$

is generated by the permutations

$$R = (1\ 3)(2\ 4\ 6\ 5), \quad S = (2\ 7)(4\ 1\ 3\ 5), \quad T = (4\ 6)(1\ 2\ 7\ 5),$$

which are cyclically permuted by $(1\ 2\ 4)(3\ 7\ 6)$. Since this transforms the

* That is, $(3, 3 \mid n, 3q; k)$.

simple group $\mathfrak{P}_1(7)$ according to an inner automorphism, (3, 3, 4; 3) is a direct product,* namely

(2.73) $$(3, 3, 4; 3) \sim \mathfrak{P}_1(7) \times G_3.$$

Putting $q=1$ in (2.71), we see that the group $(3, 3\,|\,3, n)$, of order $3n^2$, can be derived from

$$R^n = S^n = T^n = RST = TSR = 1$$

(which, by (2.24), is the direct product of two cyclic groups of order n) by adjoining an operator of period three which cyclically permutes R, S, T. (When $n=2$, this is the familiar derivation of the tetrahedral group from the four-group.) As permutations, we may write

$$\begin{aligned} R &= (1\ 4\ 7 \cdots 3n-2)(3n\ 3n-3 \cdots 6\ 3), \\ S &= (2\ 5\ 8 \cdots 3n-1)(3n-2\ 3n-5 \cdots 4\ 1), \\ T &= (3\ 6\ 9 \cdots 3n)(3n-1\ 3n-4 \cdots 5\ 2); \end{aligned}$$

whence

$$Q^{-1} = (1\ 2\ 3)(4\ 5\ 6) \cdots (\cdot\cdot\ 3n).$$

The permutations (1.63), which generate $(3, 3\,|\,3, k)$, are just Q^{-1} and $Q^{-1}R$ (with k in place of n).

By (2.71) again, $(3, 3, 3; q)$ is a subgroup of index three in $(3, 3\,|\,3, 3q)$, and (4, 4, 4; 2) in $(3, 3\,|\,4, 6)$. Hence, by (2.67),

(2.74) $$(4, 4, 4; 2)\ \textit{is infinite.}$$

2.8. **Groups of genus one.**† The infinite group $[4, 4]'$, defined by

$$R^4 = S^4 = (RS)^2 = 1,$$

has, as is well known,‡ an abelian subgroup of index four, generated by RS^{-1} and $R^{-1}S$. When represented in the usual way in the euclidean plane, these operators appear as translations. Hence every operator of $[4, 4]'$ is expressible in the form

$$S^n(RS^{-1})^p(R^{-1}S)^q.$$

The most general factor group is given by

(2.81) $$(RS^{-1})^p(R^{-1}S)^q = 1,$$

which takes the form $(RS^{-1}R^{-1}S)^q$ or $(R^2S^2)^q=1$ when $p=q$. The relation

* Since $(Q^{-1}S)^3=RTS=(1\ 5\ 2\ 6\ 3\ 7\ 4)$, the relations $Q^3=S^4=(QS)^3=(S^{-1}Q^{-1}SQ)^3=1$ imply $(Q^{-1}S)^{21}=1$. Hence $(3, 3\,|\,4, 7; 3)\sim(3, 4\,|\,3, 7; 3)\sim\mathfrak{P}_1(7)$.

† Cf. Sinkov [4].

‡ Burnside [1], p. 416.

$(R^{-1}S)^q=1$ implies $(R^2S^2)^q=1$, while $(R^2S^2)^q=1$ implies $(R^{-1}S)^{2q}=1$. In fact,* $(4, 4 \mid 2, q)$ *is the central quotient group of* $(2, 4, 4; q)$, and $(2, 4, 4; q)$ *is the central quotient group of* $(4, 4 \mid 2, 2q)$. By (2.52) and (2.51), these relationships are retained when we write "a subgroup" in place of "the central quotient group."

By (2.51) and (1.32), $(2, 4, 4; q)$ is generated by $R^{-1}SR$ and S (or by R and $S^{-1}RS$),† where

$$(2.82) \qquad \begin{cases} R = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) \cdots (\cdots 4q), \\ S = (1\ 2)(3\ 4\ 5\ 6) \cdots (\cdots 4q-2)(4q-1\ 4q). \end{cases}$$

When k is odd, (1.32) shows that $(4, 4 \mid 2, k)$ is generated by the permutations

$$(2.83) \qquad \begin{cases} R = (1\ 2\ 3\ 4) \cdots (\cdots 2k-2)(2k-1\ 2k), \\ S = (1\ 2)(3\ 4\ 5\ 6) \cdots (\cdots 2k), \end{cases}$$

which are interchanged by

$$T_1 = (1\ 2k-1)(2\ 2k)(3\ 2k-3)(4\ 2k-2) \cdots (k-2\ k+2)(k-1\ k+3).$$

Since

$$T_1S^2 = (1\ 2k-3\ 5\ 2k-7 \cdots 7\ 2k-5\ 3\ 2k-1)(2\ 2k-2\ 6\ 2k-6 \cdots 8\ 2k-4\ 4\ 2k),$$

which is of period k, S and T_1 satisfy the defining relations of $(4, 4 \mid 2, k)$; that is, T_1 transforms $(4, 4 \mid 2, k)$ according to an inner automorphism. In fact,

$$T_1 = (R^2S^2)^{(k-1)/2}R^2.$$

Now, k being odd, the central of $(4, 4 \mid 2, k)$ is of order 1. Hence $(2, 4, 4; k)$ is a direct product, namely

$$(2.84) \qquad (2, 4, 4; k) \sim (4, 4 \mid 2, k) \times G_2, \qquad k \text{ odd}.$$

We proceed to find a representation for $(2, 4, 2k; 2)$ (k unrestricted). Formula (1.31) shows that $(2k, 2k \mid 2, 2)$ is generated by the permutations

$$(2.85) \qquad \begin{cases} R = (a_1\ a_2 \cdots a_{2k})(b_1\ b_{2k})(b_2\ b_{2k-1}) \cdots (b_k\ b_{k+1}), \\ S = (b_1\ b_2 \cdots b_{2k})(a_1\ a_{2k})(a_2\ a_{2k-1}) \cdots (a_k\ a_{k+1}), \end{cases}$$

which are interchanged by

$$(2.851) \qquad T = (a_1\ b_1)(a_2\ b_2) \cdots (a_{2k}\ b_{2k}).$$

Since both R and S permute the a's among themselves, T transforms

* Sinkov [4], p. 169; Edington [1], p. 201.

† Edington (loc. cit., p. 198) gives a representation for $(4, 4 \mid 2, \alpha)$, of degree 4α. (Ours is of degree 2α.) He describes (p. 203) the group $(2, 4, 4; \beta)$, of order $8\beta^2$, but gives no representation.

$(2k, 2k \mid 2, 2)$ according to an outer automorphism. Therefore S and T generate $(2, 4, 2k; 2)$. (Other permutations of the same degree were given by Sinkov.*) When k is even, there is a central of order two which is generated by

$$(TS^k)^2 = R^k S^k = (a_1\ a_{k+1}) \cdots (a_k\ a_{2k})(b_1\ b_{k+1}) \cdots (b_k\ b_{2k}).$$

When k is odd, we can obtain a representation of degree $2k$ (instead of $4k$), by deriving $(2, 4, 2k; 2)$ from $(4, 4 \mid 2, k)$. Using the generating permutations (2.83) for $(4, 4 \mid 2, k)$ (k odd), we seek a permutation which will transform R, S into S^{-1}, R^{-1}. Such is clearly

(2.86) $$T' = (1\ 2k)(2\ 2k-1) \cdots (k\ k+1).$$

This transforms $(4, 4 \mid 2, k)$ according to an outer automorphism, since ST' is of period $2k$ although $(4, 4 \mid 2, k)$ contains no operator whose period is greater than k. Hence $(2, 4, 2k; 2)$ (k odd) is generated by T' and

(2.861) $$S = (1\ 2)(3\ 4\ 5\ 6) \cdots (\cdots\ 2k).$$

The infinite group

$$R^3 = S^3 = (RS)^3 = 1$$

is closely analogous to $[4, 4]'$. It has† an abelian subgroup of index three, generated by RS^{-1} and $R^{-1}S$. When represented in the usual way in the euclidean plane, these operators appear as translations. The most general factor group is again given by (2.81), which takes the form $(RS^{-1}R^{-1}S)^q = 1$ when $p = q$. The relation $(R^{-1}S)^q = 1$ implies $(RS^{-1}R^{-1}S)^q = 1$, while $(RS^{-1}R^{-1}S)^q = 1$ implies $(R^{-1}S)^{3q} = 1$. In fact,‡ $(3, 3 \mid 3, q)$ *is the central quotient group of* $(3, 3, 3; q)$, and $(3, 3, 3; q)$ *is the central quotient group of* $(3, 3 \mid 3, 3q)$. By (2.72) and (2.71), these relationships are retained when we write "a subgroup" in place of "the central quotient group."

By (2.71) ($n = 3$) and (1.63), $(3, 3, 3; q)$ is generated by QSQ^{-1} and S, where

$$Q = (1\ 2\ 3)(4\ 5\ 6) \cdots (9q-2\ 9q-1\ 9q),$$
$$S = (9q\ 1\ 2)(3\ 4\ 5) \cdots (9q-3\ 9q-2\ 9q-1).$$

In the form

$$R^3 = S^3 = T^3 = RST = (R^{-1}S)^k = (S^{-1}T)^k = (T^{-1}R)^k = 1,$$

$(3, 3 \mid 3, k)$ is generated by

* Sinkov [2], p. 79.

† Burnside [1], p. 414.

‡ Sinkov [4], p. 168; Edington [1], p. 210.

$$(2.87)\quad \begin{cases} R = (1\ 2\ 3)(4\ 5\ 6)\cdots(3k-2\ 3k-1\ 3k), \\ S = (3k\ 1\ 2)(3\ 4\ 5)\cdots(3k-3\ 3k-2\ 3k-1), \\ T = (3k-1\ 3k\ 1)(2\ 3\ 4)\cdots(3k-4\ 3k-3\ 3k-2). \end{cases}$$

When $k\equiv 0 \pmod 3$, there is no permutation of $1, 2, \cdots, 3k$ which will cyclically permute R, S, T. When $k\equiv \pm 1 \pmod 3$, R, S, T are permuted by Q_1^{-1}, where

$$Q_1^{\pm 1} = (1\ k+1\ 2k+1)(2\ k+2\ 2k+2)\cdots(k\ 2k\ 3k).$$

Clearly

$$Q_1^{-1}S = (1\ 2k+2\ k+3\ 4\cdots 2k-1\ k)(k+1\ 2\ 2k+3\ k+4\cdots \\ 3k-1\ 2k)(k-1\ 3k-3\ 2k-5\cdots k+5\ 3\ 2k+1)$$

or

$$(1\ k+2\ 2k+3\ 4\cdots k-1\ 2k)(2k+1\ 2\ k+3\ 2k+4\cdots \\ 3k-1\ k)(2k-1\ 3k-3\ k-5\cdots 2k+5\ 3\ k+1)$$

according as $k\equiv 1$ or $-1 \pmod 3$. Since this is of period k, Q_1 and S satisfy the defining relations of $(3, 3\,|\,3, k)$; that is, Q_1 transforms $(3, 3\,|\,3, k)$ according to an inner automorphism. Since k is not divisible by 3, the central of $(3, 3\,|\,3, k)$ is of order 1. Hence $(3, 3, 3; k)$ is a direct product,* namely

$$(2.88)\qquad (3, 3, 3; k) \sim (3, 3\,|\,3, k)\times G_3, \qquad k\not\equiv 0 \pmod 3.$$

The generators R, S of $(3, 3\,|\,3, k)$ ((1.63) or (2.87)) are interchanged by either of the permutations

$$T = (3\ 3k)(4\ 3k-2)(5\ 3k-1)(6\ 3k-3)(7\ 3k-5)(8\ 3k-4)\cdots,$$

$$T_1 = (1\ 3k-2)(2\ 3k-1)(3\ 3k-3)(4\ 3k-5)(5\ 3k-4)(6\ 3k-6)\cdots.$$

The permutation T transforms $(3, 3\,|\,3, k)$ according to an outer automorphism,† since ST is of period six although $(3, 3\,|\,3, k)$ contains, in general, no operator of that period. Hence, by (2.52), $(2, 3, 6; k)$ is generated‡ by T, and

$$(2.89)\qquad S = (3k\ 1\ 2)(3\ 4\ 5)\cdots(\cdot\cdot\ 3k-1).$$

T can be replaced by T_1 whenever $k>2$; but when $k=2$, ST_1 is of period three instead of six, showing that $(2, 3, 6; 2)$ is the direct product of the tetrahedral group and the group of order two, that is, the *pyritohedral* group.

---

* See Miller [2], p. 668, for the case when $k=2$.

† This can also be seen by trying to express T in the form $S^n(RS^{-1})^p(R^{-1}S)^q$.

‡ Cf. Edington [1], p. 205, where a representation of degree $6k$ is given.

On the other hand, R^{-1} and S are interchanged by

(2.891) $$T' = (1\ 3k-1)(2\ 3k-2)(3\ 3k-3)\cdots.$$

This and S are equivalent to Sinkov's generators* for $(2, 3, 2k; 3)$.

2.9. **An extension of the criterion.** Since $(l, m, n; q)$ involves l, m, n symmetrically, Theorem D suggests that possibly $(l, m, n; q)$ is finite (or collapses) whenever

(2.91) $$\cos 2\pi/l + \cos 2\pi/m + \cos 2\pi/n + (1 + 4\cos\pi/l\cos\pi/m\cos\pi/n)\cos\pi/q < 0.\dagger$$

This inequality, with $l=2$, is just (2.61). The solutions with $2<l\leqq m\leqq n$ and $q>1$ are

$$(3, 3, 3; q),\quad (3, 3, n; 2),\quad (3, 3, 4; 3),\quad (3, 4, 4; 2),\quad (3, 4, 5; 2).$$

The first and third of these have already been described.

Sinkov has shown‡ that $(3, 3, n; 2)$ collapses unless $n=2, 3, 6$, or 12. The relations

$$R^3 = S^3 = (R^{-1}S^{-1}RS)^2 = 1$$

suffice to define $(3, 3, 12; 2)$, of order 288. As a rotation group in four dimensions§ this is denoted by $[3, 4, 3]''$. It is a subgroup of index four in $(4, 8\mid 2, 3)$. Its central quotient group $(3, 3, 6; 2)$ is the direct product of two tetrahedral groups.|| By applying the method of §2.7 to the group

$$R^3 = S^3 = T^3 = (R^{-1}S)^2 = (S^{-1}T)^2 = (T^{-1}R)^2 = 1,$$

we find

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}'$$

or $[3^{1,1,1}]'$ (of order 96)¶ to be an invariant subgroup in $[3, 4, 3]''$.

---

* Sinkov [2], p. 82.

† The term $4\cos\pi/l\cos\pi/m\cos\pi/n\cos\pi/q$ was added on June 27, 1938. Without it the criterion would admit $(3, 3, 3; \infty)$ and $(3, 3, 4; 4)$. A proof that the latter group is infinite will be published elsewhere. The proviso $q>1$ has to be inserted because the criterion would admit $(l, \infty, \infty; 1)$.

‡ Sinkov [2], pp. 76–78.

§ Coxeter [5], pp. 68–70. The subgroup

$$R^{12} = S^{12} = T^{12} = RST = (R^{-1}S)^2 = (S^{-1}T)^2 = (T^{-1}R)^2 = 1,$$

of order 96, is the rotation group whose central quotient group is no. XIV ($m=2$) of Goursat [1], p. 65.

|| Goursat's no. XX.

¶ Coxeter [1], p. 149, (16.75), $n=p=q=1$, $h_0=i_0=j_0=3$. This is the subgroup D of Sinkov [2], p. 76.

The relations

$$R^4 = S^4 = (RS)^3 = (R^{-1}S^{-1}RS)^2 = 1,$$

which define (3, 4, 4; 2), are satisfied by the permutations

$$R = (1\ 2\ 3\ 4)(7\ 8\ 9\ 10)(6\ 12),$$
$$S = (4\ 5\ 6\ 7)(3\ 8\ 11\ 12)(1\ 9),$$

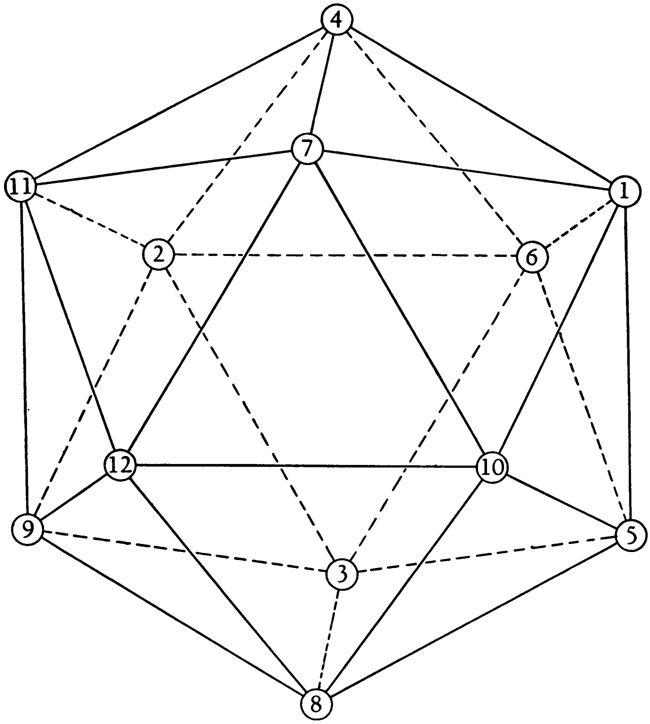

Fig. 1
The icosahedron

which may be regarded as operating on the vertices of an icosahedron, as in Fig. 1. Although R and S are not themselves symmetries of the icosahedron, the combinations

$$RS = (1\ 2\ 8)(3\ 5\ 6)(4\ 9\ 10)(7\ 11\ 12),$$
$$S^{-1}R = (1\ 10\ 7\ 12\ 11\ 9\ 2\ 3\ 6\ 5)(4\ 8)$$

are, respectively, a rotation and a rotary reflection; these generate the extended icosahedral group [3, 5], which is the direct product of the group of order two generated by

$$I = (R^{-1}S)^5 = (S^{-1}R)^5 = (1\ 9)(2\ 10)(3\ 7)(4\ 8)(5\ 11)(6\ 12)$$

and the icosahedral group generated by RS and

$$(R^{-1}S)^4 = S^{-1}RI.$$

Thus the order of $(3, 4, 4; 2)$ is at least 240. The fact that it is exactly 240 may be established by enumerating the twenty cosets of the tetrahedral subgroup generated by $R^{-1}S^{-1}$ and RS.

The operator I interchanges pairs of opposite vertices of the icosahedron, and generates the central, of order two. The quotient group,* of order 120, is generated by the permutations

$$(1\ 2\ 3\ 4), \qquad (3\ 4\ 5\ 6).$$

It is $G_{5!}$, since the relations

$$R^4 = S^4 = (RS)^3 = (R^{-1}S^{-1}RS)^2 = (R^{-1}S)^5 = 1$$

are also satisfied by

$$(1\ 2\ 3\ 4), \qquad (5\ 4\ 3\ 2).$$

To sum up, the group $(3, 4, 4; 2)$, of order 240, has $G_{5!/2} \times G_2$ as a subgroup, and has a central of order two whose quotient group is $G_{5!}$; but it is not $G_{5!} \times G_2$ (since R^2S is of period twelve).

Finally, $(3, 4, 5; 2)$ is isomorphic with the alternating group of degree six, as generated by

(2.92) $$R = (1\ 4)(2\ 6\ 3\ 5), \qquad S = (1\ 2\ 3\ 4\ 5).$$

By enumerating the thirty cosets of the tetrahedral subgroup generated by $R^{-1}S^{-1}$ and RS, we easily find that the order is just 360. Hence

(2.93) $$(3, 4, 5; 2) \sim G_{6!/2}.$$

Theorem D can thus be extended as follows:

THEOREM D′. *For all infinite groups* $(l, m, n; q)$ $(q > 1)$,

$$\cos 2\pi/l + \cos 2\pi/m + \cos 2\pi/n + (1 + 4 \cos \pi/l \cos \pi/m \cos \pi/n) \cos \pi/q \geqq 0.$$

We observe that this expression vanishes for the infinite groups $(3, 3, 3; \infty)$, $(4, 4, 4; 2)$.

All the known finite groups $(l, m, n; q)$ are collected together in Table II, at the end of the paper.

* $(4, 4 \mid 3, 5; 2)$.

CHAPTER III. $G^{m,n,p}$

3.1. The derivation of $G^{m,n,p}$ (p even) from its subgroup $(2, m, n; p/2)$. $G^{m,n,p}$ means the group defined by

$$(3.11) \qquad A^m = B^n = C^p = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1.$$

This is symmetrical between m, n, p: for cyclic permutation, obviously; and for transposition, by changing A, B, C into C^{-1}, B^{-1}, A^{-1}, respectively. Since

$$ABCABC = B^{-1}A^{-1}\cdot A^{-1}C^{-1}\cdot C^{-1}B^{-1} = (BC^2A^2B)^{-1},$$

the defining relations imply

$$(3.12) \qquad A^2B^2C^2 = 1.$$

Hence, if any one of m, n, p is odd, the corresponding one of A, B, C is expressible in terms of the other two. Thus $G^{m,n,p}$ (p odd) is a factor group of $(2, m, n; p)$.

If p is even, all the relations (3.11) involve C an even number of times; hence the set of all operators which involve C an even number of times is a subgroup of index two. Since

$$CA = A^{-1}C^{-1}, \quad C^{-1}A^{-1} = AC, \quad CB = B^{-1}C^{-1}, \quad C^{-1}B^{-1} = BC, \quad C^2 = B^{-2}A^{-2},$$

this subgroup is generated by A and B, which satisfy

$$(3.13) \qquad A^m = B^n = (AB)^2 = (A^2B^2)^{p/2} = 1.$$

Actually, these last relations completely define the subgroup. This fact will emerge from the following more general investigation.

The group $(l, m, n; q)$, in the form

$$S^m = T^n = (ST)^l = (S^{-1}T^{-1}ST)^q = 1,$$

clearly possesses an automorphism which replaces the generators by their inverses. If we adjoin an involutory operator R_2 which transforms the group according to this automorphism, we obtain a larger group, say $((l, m, n; 2q))$, defined by

$$(3.14) \qquad R_2^2 = S^m = T^n = (R_2S)^2 = (R_2T)^2 = (ST)^l = (R_2ST)^{2q} = 1.$$

To see that this involves l, m, n symmetrically, we define

$$R_1 = SR_2, \qquad R_3 = R_2T.$$

Eliminating S and T, we obtain

$$(3.15) \qquad R_1^2 = R_2^2 = R_3^2 = (R_3R_1)^l = (R_1R_2)^m = (R_2R_3)^n = (R_1R_2R_3)^{2q} = 1.$$

This group, then, contains $(l, m, n; q)$ as a subgroup of index two. Writing the latter in the form (2.111), we see that

$$R = R_3R_1, \qquad S = R_1R_2, \qquad T = R_2R_3.$$

Thus each of R_1, R_2, R_3 transforms two of R, S, T into their inverses.

By defining $U = R_3R_2R_1$, we may write (3.15) in the form

$$S^m = T^n = U^{2q} = (ST)^l = (TU)^2 = (US)^2 = (STU)^2 = 1. \tag{3.16}$$

Putting $l = 2$, we have

$$((2, m, n; 2q)) \sim G^{m,n,2q}.$$

Thus $(2, m, n; q)$ is a subgroup of index two in $G^{m,n,2q}$, and the latter can be derived from the former by adjoining an involutory operator which transforms the (two) generators into their inverses.

When $(2, m, n; q)$ is given in the form

$$R^m = S^n = (RS)^2 = (R^2S^2)^q = 1,$$

we naturally write $A = R$, $B = S$; and the involutory operator which transforms these into their inverses is BCA. (For $BCA \cdot ABCA = BCABC = A^{-1}$, and $BCAB \cdot BCA = CABCA = B^{-1}$.) On the other hand, when the same group is given in the form

$$S^n = T^2 = (ST)^m = (S^{-1}TST)^q = 1,$$

we write $A = TS^{-1}$, $B = S$; and the involutory operator which transforms S into its inverse, while leaving T $(=AB)$ unchanged, is CA.

3.2. **The extended polyhedral groups.** The extended polyhedral group $[m, n]$ or*

is defined by

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^m = (R_2R_3)^n = (R_3R_1)^2 = 1. \tag{3.21}$$

If we write

$$A = R_1R_2, \qquad B = R_2R_3, \qquad C = R_3R_2R_1,$$

so that

$$R_1 = BC, \qquad R_2 = BCA, \qquad R_3 = CA,$$

this definition becomes

$$A^m = B^n = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1. \tag{3.22}$$

---

* Coxeter [2], pp. 589, 619.

Hence

$$[m, n] \sim G^{m,n,p},$$

where p, the period of C, remains to be determined.

This is a "group of genus zero."* If we force p to have a smaller value than its "natural" value, we obtain the general group $G^{m,n,p}$, which is a factor group of $[m, n]$. When $1/m+1/n \leqq 1/2$, the "natural" value is infinite, and

$$[m, n] \sim G^{m,n,\infty}.$$

For the present, however, we concentrate our attention on the case when $1/m+1/n>1/2$.

Since each of the relations (3.21) involves an even number of R's, the period of $R_3R_2R_1$, deduced as a consequence of those relations, must be even. The ordinary (unextended) polyhedral group $[m, n]'$ is generated by the rotations A and B; hence

$$[m, n]' \sim (2, m, n; p/2).$$

Using the values of q $(q=p/2)$ that were found in §2.3, we see that the (finite) extended polyhedral groups are

$$G^{2,n,2n}\ (n \text{ odd}), \quad G^{2,n,n}\ (n \text{ even}),$$

$$G^{3,3,4}, \quad G^{3,4,6}, \quad G^{3,5,10}.$$

The last four of these are covered by the formula

$$\cos 2\pi/p = 1 + \cos 2\pi/m + \cos 2\pi/n. \tag{3.23}$$

In §4.7, we shall see that this formula has a geometrical significance, in spite of its failure when $m=2$ and n is odd.

Since $G^{2,n,p}$ has a subgroup $(2, n, p; 1)$, (2.22) and (2.23) show that the extended dihedral groups

$$\left\{\begin{matrix} G^{2,n,2n} & (n \text{ odd}) \\ G^{2,n,n} & (n \text{ even}) \end{matrix}\right\} \sim [n] \times G_2 \tag{3.24}$$

are the *only* groups of this form; any other values for p (given n) would cause collapse. It is clear, also, that $G^{3,3,p}$, $G^{3,4,p}$, $G^{3,5,p}$ must collapse whenever p is not a divisor of 4, 6, 10, respectively; and we have just seen that they collapse when $p=2$. Apart from the extended polyhedral groups themselves, there remains only $G^{3,5,5}$. This, being a proper factor group of the extended

* Dyck [1], p. 34.

icosahedral group $G^{3,5,10}$, must be the icosahedral group itself.

The following generating permutations are easily verified:

	$G^{3,3,4} \sim G_{4!}$	$G^{3,4,6} \sim G_{4!} \times G_2$	$G^{3,5,5} \sim G_{5!/2}$	$G^{3,5,10} \sim G_{5!/2} \times G_2$
A	(3 2 1)	(4 3 2)	(5 3 2)	(5 3 2)
B	(4 3 2)	(1 2 3 4)	(1 2 3 4 5)	(1 2 3 4 5)
C	(1 2 3 4)	(3 2 1)(5 6)	(1 2 4 3 5)	(1 2 4 3 5)(6 7)

3.3. **Cases in which m, n are odd, while p is even.** By (2.53), the group (2, 5, 5; 2), of order 80, is generated (in the form (3.13)) by

$$A = (0\ 1\ 2\ 3\ 4)(5\ 6\ 7\ 8\ 9), \qquad B = (0\ 9\ 8\ 7\ 6)(5\ 4\ 3\ 2\ 1).$$

These are transformed into their inverses by

$$(0\ 5)(1\ 9)(2\ 8)(3\ 7)(4\ 6),$$

which, being an odd permutation,* may be identified with BCA. Since B is of odd period, the whole group $G^{5,5,4}$ (or $G^{4,5,5}$) is generated by A and

$$C = (0\ 3\ 5\ 8)(1\ 7)(2\ 6).$$

But these permutations satisfy

$$C^4 = A^5 = (CA)^2 = (C^{-1}A)^4 = 1.$$

Hence

$$G^{4,5,5} \sim (4, 5 \mid 2, 4). \tag{3.31}$$

This may be compared with the relation

$$G^{4,3,3} \sim (4, 3 \mid 2, 4),$$

which expresses the well known simple isomorphism between the extended tetrahedral and unextended octahedral groups.

In other cases we suppose $(2, m, n; q)$ to be given in the form

$$S^n = T^2 = (ST)^m = (S^{-1}TST)^q = 1,$$

where $S=B$, $T=AB$, and we derive $G^{m,n,2q}$ by adjoining the operator CA which transforms S into its inverse, leaving T unchanged. We use a tabular arrangement, as in §2.5. ($G^{3,3,4}$ and $G^{3,5,10}$ have already been described; the latter is included now for the sake of comparison.)

* The even permutation (1 4)(2 3)(6 9)(7 8) could have been used just as well, but it is less obviously outside the group (2, 5, 5; 2).

$(2, m, n; q)$	$(2, 3, 5; 5) \sim G_{5!/2}$	$(2, 3, 7; 4) \sim \mathfrak{P}_1(7)$
S T	(1 2 3 4 5) (1 2)(4 5)	(0 1 2 3 4 5 6) (0 ∞)(1 6)(2 3)(4 5)
Automorphism (S, S^{-1})	(1 5)(2 4) (inner)	(1 6)(2 5)(3 4) (outer)
$G^{m,n,2q}$	$G^{3,5,10} \sim G_{5!/2} \times G_2$	$G^{3,7,8} \sim \tilde{\mathfrak{P}}_1(7)$

$(2, m, n; q)$	$(2, 3, 7; 6) \sim \mathfrak{P}_1(13)$	$(2, 3, 7; 7) \sim \mathfrak{P}_1(13)$
S T	(0 10 2 9 3 8 5)(1 7 11 4 6 ∞ 12) (1 12)(2 11)(3 10)(4 9)(5 8)(6 7)	(0 9 11 5 12 1 10)(2 7 6 4 3 8 ∞) (1 12)(2 11)(3 10)(4 9)(5 8)(6 7)
Automorphism (S, S^{-1})	(0 ∞)(6 10)(2 4)(9 11)(3 7)(1 8)(5 12) (outer)	(0 ∞)(8 9)(3 11)(4 5)(6 12)(1 7)(2 10) (outer)
$G^{m,n,2q}$	$G^{3,7,12} \sim \tilde{\mathfrak{P}}_1(13)$	$G^{3,7,14} \sim \tilde{\mathfrak{P}}_1(13)$

$(2, m, n; q)$	$(2, 5, 9; 2) \sim \mathfrak{P}_1(19)$
S T	(1 16 9 11 5 4 7 17 6)(2 13 18 3 10 8 14 15 12) (0 7)(1 ∞)(2 14)(3 17)(4 18)(5 9)(6 15)(8 11)(10 13)(12 16)
Automorphism (S, S^{-1})	(0 ∞)(1 7)(4 16)(5 9)(6 17)(3 15)(10 14)(12 18)(2 13) (outer)
$G^{m,n,2q}$	$G^{4,5,9} \sim \tilde{\mathfrak{P}}_1(19)$

$(2, m, n; q)$	$(2, 3, 11; 4) \sim \mathfrak{P}_1(23)$
S T	(1 9 12 16 6 8 3 4 13 2 18)(5 22 14 11 7 17 15 20 19 10 21) (0 3)(1 ∞)(2 22)(4 8)(5 12)(6 19)(7 16)(9 18)(10 11)(13 20)(14 15)(17 21)
Automorphism (S, S^{-1})	(0 ∞)(1 3)(4 18)(2 13)(8 9)(6 12)(5 19)(10 21)(20 22)(14 15)(11 17) (outer)
$G^{m,n,2q}$	$G^{3,8,11} \sim \tilde{\mathfrak{P}}_1(23)$

The generating permutations are taken from (2.39), (2.391), (2.64), (2.66). In each case they are even permutations; so whenever the desired automorphism is effected by an odd permutation, we can be sure that it is an outer automorphism. (This permutation is easily found by observing which symbols are left unchanged by one of the generators.)

The apparently possible groups

$$G^{3,6,n}\ (n \text{ odd}),\quad G^{3,7,10},\quad G^{3,8,9},\quad G^{4,4,n}\ (n \text{ odd}),\quad G^{4,5,7}$$

collapse with their would-be subgroups (see §2.6, §2.3, and the Appendix)

$$(2, 3, n; 3),\qquad (2, 3, 7; 5),\qquad (2, 3, 9; 4),\qquad (2, 4, n; 2),\qquad (2, 5, 7; 2).$$

3.4. **Groups of genus one: $G^{4,4,2k}$ and $G^{3,6,2k}$.** The general results of §2.5 and §3.1 are illustrated as "genealogies" in Fig. 2, where each single-headed arrow stands for "is a subgroup of index two in," while each double-headed arrow stands for "is a subgroup of index three in."

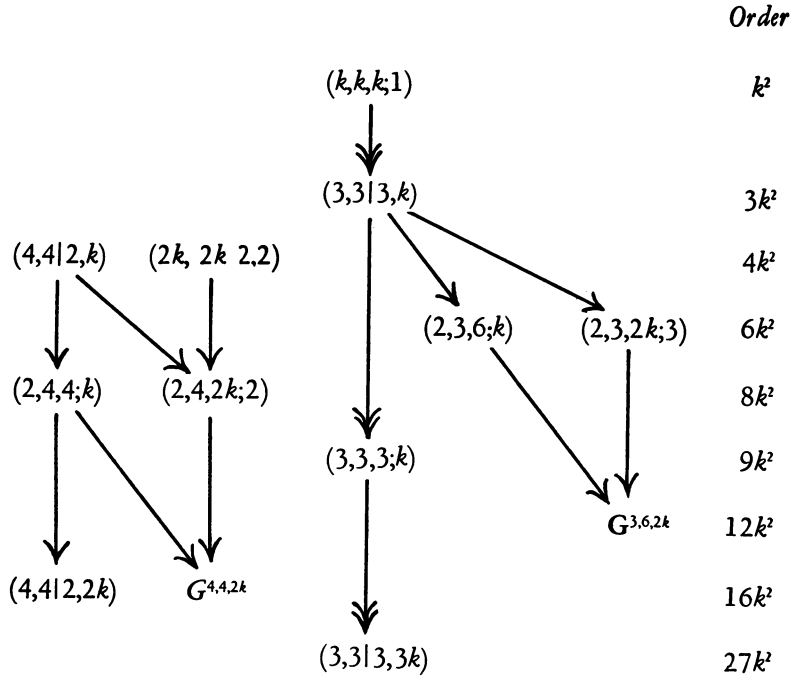

FIG. 2
Groups of genus one

When k is odd, (2.86) and (2.861) show that $(2, 4, 2k; 2)$ is generated by

$$AB = (1\ 2k)(2\ 2k-1)\cdots(k\ k+1),$$
$$B = (1\ 2)(3\ 4\ 5\ 6)\cdots(2k-3\cdot\cdot 2k).$$

B is transformed into its inverse, and AB into itself, by

$$(1\ 2)(3\ 4)(5\ 6)\cdots(2k-1\ 2k).$$

Calling this C_1A, where

$$A=(1\ 2k-1)(3\ 2k-3)\cdots(k-2\ k+2)(2\ 2k-2\ 6\ 2k-6\cdots 2k-4\ 4\ 2k),$$

we find

$$C_1=(1\ 2k)(2\ 2k-1\ 4\ 2k-3)(3\ 2k-4\ 5\ 2k-2)(6\ 2k-5\ 8\ 2k-7)\cdots.$$

Since $C_1^{-1}B$ is of period k, C_1 and B generate $(4, 4\,|\,2, k)$, whereas the C and B of $G^{2k,4,4}$ generate the larger group $(2, 4, 4; k)$. Thus $(2, 4, 2k; 2)$ is transformed according to an inner automorphism. Now, k being odd, the central of $(2, 4, 2k; 2)$ is of order 1. Hence $G^{2k,4,4}$ is a direct product, namely

$$G^{4,4,2k} \sim (2, 4, 2k; 2) \times G_2, \qquad k \text{ odd.*} \tag{3.41}$$

By (2.85) and (2.851), on the other hand, whatever be the parity of k, $(2, 4, 2k; 2)$ is generated by

$$B = (a_1\ a_{2k})(a_2\ a_{2k-1})\cdots(a_k\ a_{k+1})(b_1\ b_2\cdots b_{2k}),$$
$$AB = (a_1\ b_1)(a_2\ b_2)\cdots(a_{2k}\ b_{2k}).$$

B is transformed into the inverse, and AB into itself, by

$$(a_1\ a_{2k})(a_2\ a_{2k-1})\cdots(a_k\ a_{k+1})(b_1\ b_{2k})(b_2\ b_{2k-1})\cdots(b_k\ b_{k+1}).$$

Calling this CA, where

$$A = (a_1\ b_{2k})(b_1\ a_{2k}\ b_{2k-1}\ a_2)(b_2\ a_{2k-1}\ b_{2k-2}\ a_3)\cdots(b_{k-1}\ a_{k+2}\ b_{k+1}\ a_k)(a_{k+1}\ b_k),$$

we find

$$C = (a_1\ b_1)(a_2\ b_2\ a_{2k}\ b_{2k})(a_3\ b_3\ a_{2k-1}\ b_{2k-1})\cdots(a_k\ b_k\ a_{k+2}\ b_{k+2})(a_{k+1}\ b_{k+1}).$$

Since $A^{-1}C=(a_1\ a_2\cdots a_{2k})(b_1\ b_2\cdots b_{2k})$, A and C generate the whole group $(2, 4, 4; k)$, and therefore A, B, C generate $G^{4,2k,4}$ (that is, $G^{4,4,2k}$).

Similarly, (2.89) and (2.891) show that $(2, 3, 2k; 3)$ is generated by

$$B = (3k\ 1\ 2)(3\ 4\ 5)\cdots(3k-3\ 3k-2\ 3k-1),$$
$$AB = (1\ 3k-1)(2\ 3k-2)(3\ 3k-3)\cdots.$$

B is transformed into its inverse, and AB into itself, by

$$(1\ 2)(4\ 5)\cdots(3k-2\ 3k-1).$$

Calling this CA, where

* Cf. (2.84).

$$A = (3k\ 2\ 3k-3\ 5\ 3k-6\ 8 \cdots 3\ 3k-1)(1\ 3k-2)(4\ 3k-5)(7\ 3k-8)\cdots,$$

we find

$$C = (1\ 3k\ 3k-1)(2\ 3k-2\ 3\ 3k-4\ 4\ 3k-3)(5\ 3k-5\ 6\ 3k-7\ 7\ 3k-6)\cdots.$$

Since $(2, 3, 2k; 3)$ has, in general, no operator of period six,* we can be sure that A, B, C (or just A and C, since the period of B is odd) generate $G^{2k,3,6}$ (that is, $G^{3,6,2k}$).

3.5. **The derivation of $G^{m,2n,2k}$ from its subgroup $(m, m \mid n, k)$.** We saw, in §2.5, that the groups $(2, m, 2n; k)$, $(2, m, 2k; n)$ can be derived from $(m, m \mid n, k)$ by adjoining involutory operators T, T', respectively; T interchanges R, S, while T' transforms each into the inverse of the other. The product TT' (or $T'T$) transforms each into its own inverse (see §1.2). Let us now adjoin these two permutable operators simultaneously, that is, adjoin the four-group which they generate.

Along with the relations

$$R^m = S^m = (RS)^n = (R^{-1}S)^k = 1,$$

which define $(m, m \mid n, k)$, we now have

$$T^2 = T'^2 = 1, \qquad TT' = T'T, \qquad TST = R = T'S^{-1}T'.$$

Eliminating R, we obtain

$$S^m = T^2 = T'^2 = (ST)^{2n} = (ST')^{2k} = (TT')^2 = (STT')^2 = 1.$$

We can identify this with $G^{m,2n,2k}$ by writing

$$(3.51)\qquad \begin{array}{lll} S = A^{-1}, & T = AB, & T' = CA; \\ A = S^{-1}, & B = ST\ (= TR), & C = T'S\ (= R^{-1}T'). \end{array}$$

Thus $(m, m \mid n, k)$ is an invariant subgroup of index four in $G^{m,2n,2k}$.

When

$$(2, m, 2n; k) \sim (2, m, 2k; n)$$

(in particular, when $n=k$), it may happen that TT' transforms $(m, m \mid n, k)$ according to an inner automorphism. Then T' transforms $(2, m, 2n; k)$ according to an inner automorphism, and, if the order of the central of $(m, m \mid n, k)$ is odd,

$$(3.52)\qquad G^{m,2n,2k} \sim (2, m, 2n; k) \times G_2.$$

Applying this principle to $(5, 5 \mid 2, 3)$, $(3, 3 \mid 4, 4)$, $(3, 3 \mid 4, 5)$ (see (2.56)), we obtain

* See Coxeter [5], p. 67, where such an operator would appear as a hexagonal rotation.

(3.53) $$G^{4,5,6} \sim G_{5!} \times G_2,$$

(3.54) $$G^{3,8,8} \sim \mathfrak{P}_1(7) \times G_2,$$

(3.55) $$G^{3,8,10} \sim (2, 3, 8; 5) \times G_2.$$

When the groups $(2, m, 2n; k)$, $(2, m, 2k; n)$ are distinct, it may happen that T and T' transform $(m, m \mid n, k)$ according to outer and inner automorphisms, respectively. Then T' transforms $(2, m, 2n; k)$ according to an inner automorphism, and (with the usual proviso about the central) (3.52) follows again.

Applying this principle to $(5, 5 \mid 4, 2)$* and $(7, 7 \mid 2, 3)$, we obtain

(3.56) $$G^{4,5,8} \sim (2, 5, 8; 2) \times G_2,$$

(3.57) $$G^{4,6,7} \sim \mathfrak{P}_1(13) \times G_2.$$

The same principle, applied to $(4, 4 \mid k, 2)$, provides an alternative proof for (3.41).

3.6. **Definitions for $G^{m,n,p}$ (m odd) in terms of two generators.** When m is odd, the relation (3.12) gives

$$A = A^{-(m-1)} = (B^2C^2)^{(m-1)/2}.$$

Thus $G^{m,n,p}$ is generated by B and C. By direct substitution, (3.11) becomes

$$B^n = C^p = (BC)^2 = (B^2C^2)^m = [(B^2C^2)^{(m-1)/2}B]^2$$
$$= [C(B^2C^2)^{(m-1)/2}]^2 = 1,$$
$$[C(B^2C^2)^{(m-1)/2}B]^2 = 1.$$

The last relation is superfluous, since $(BC)^2=1$ implies

$$\begin{aligned} C(B^2C^2)^{(m-1)/2}B &= CB\cdot BC\cdot \cdots \cdot BC\cdot CB \\ &= B^{-1}C^{-1}\cdot C^{-1}B^{-1}\cdot \cdots \cdot C^{-1}B^{-1}\cdot B^{-1}C^{-1} \\ &= B^{-1}(C^{-2}B^{-2})^{(m-1)/2}C^{-1}. \end{aligned}$$

We may also omit any one of the three relations

$$(B^2C^2)^m = 1, \qquad [(B^2C^2)^{(m-1)/2}B]^2 = 1, \qquad [C(B^2C^2)^{(m-1)/2}]^2 = 1,$$

since $(BC)^2=1$ implies

$$\begin{aligned} [(B^2C^2)^{(m-1)/2}B]^2[C(B^2C^2)^{(m-1)/2}]^2 &= (B^2C^2)^{(m-1)/2}B\cdot BC\cdot C(B^2C^2)^{(m-1)/2} \\ &= (B^2C^2)^m. \end{aligned}$$

Thus $G^{m,n,p}$ (m odd) is defined by

(3.61) $$B^n = C^p = (BC)^2 = (B^2C^2)^m = [B(B^2C^2)^{(m-1)/2}]^2 = 1$$

* Or to $(5, 5 \mid 2, 4)$, interchanging the roles of T and T'.

or

$$(3.611)\qquad B^n = C^p = (BC)^2 = [B(B^2C^2)^{(m-1)/2}]^2 = [(B^2C^2)^{(m-1)/2}C]^2 = 1.$$

Of these two definitions, the latter is more symmetrical (especially when $n=p$); but the former is simpler, and exhibits $G^{m,n,p}$ as a factor group of $(2, n, p; m)$. In particular, putting $m=3$, we see that $G^{3,n,p}$ is defined by

$$(3.62)\qquad B^n = C^p = (BC)^2 = (B^2C^2)^3 = (B^3C^2)^2 = 1$$

or by

$$(3.621)\qquad B^n = C^p = (BC)^2 = (B^3C^2)^2 = (B^2C^3)^2 = 1.$$

In order to conform with Sinkov's notation,* let us write $B=P$, $C^2=Q$. When p is even, we obtain the subgroup $(2, m, n; p/2)$ in the form

$$(3.63)\qquad P^n = Q^{p/2} = (P^2Q)^m = [P(P^2Q)^{(m-1)/2}]^2 = 1.$$

To see that this definition is sufficient, we put $(P^2Q)^{(m-1)/2}=S$, so that $Q=P^{-2}S^{-2}$, and deduce

$$S^m = P^n = (SP)^2 = (S^2P^2)^{p/2} = 1.$$

When p (as well as m) is odd, so that $C=C^{p+1}=Q^{(p+1)/2}$, (3.61) becomes

$$(3.64)\qquad P^n = Q^p = (P^2Q)^m = (PQ^{(p+1)/2})^2 = [P(P^2Q)^{(m-1)/2}]^2 = 1.$$

This exhibits $G^{m,n,p}$ as a factor group of $(2, m, n; p)$ (see (3.63)), with the very simple extra relation

$$(PQ^{(p+1)/2})^2 = 1.$$

Putting $m=3$, we have†

$$(3.65)\qquad P^n = Q^p = (P^2Q)^3 = (P^3Q)^2 = 1$$

for $(2, 3, n; p)$, and

$$(3.66)\qquad P^n = Q^p = (P^2Q)^3 = (P^3Q)^2 = (PQ^{(p+1)/2})^2 = 1$$

for $G^{3,n,p}$ (p odd).

In (3.66), any one of the three expressions

$$(P^2Q)^3, \qquad (P^3Q)^2, \qquad (PQ^{(p+1)/2})^2$$

may be replaced by $(P^2Q^{(p+3)/2})^2$; the first two, because

$$(B^2C^2B)^2(CB^2C^2)^2 = (B^2C^2)^3$$

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* Sinkov [3], p. 68. In "$Q=C^2$, $P=A$," A is a misprint for B.

† For the case when $n=7$, see Brahana [2], p. 349. Our P (or B) is his R.

(as we saw above, before putting $m=3$), and the last by the following argument, due to Sinkov. He has shown* that the relations $(P^2Q)^3=(P^3Q)^2=1$ imply $(PQP^2Q^\alpha)^2=1$ (for any α). If also $(P^2Q^\alpha)^2=1$, where $\alpha=(p+3)/2$, we have

$$(PQ^{(p+1)/2})^2 = (P^{-1}\cdot P^2Q^\alpha\cdot Q^{-1})^2 = (P^{-1}Q^{-\alpha}P^{-2}Q^{-1})^2 = (QP^2Q^\alpha P)^{-2} = 1.$$

Thus $G^{3,n,p}$ (p odd) is defined by

$$P^n = Q^p = (P^2Q)^3 = (P^3Q)^2 = (P^2Q^{(p+3)/2})^2 = 1. \tag{3.661}$$

This is the definition used by Sinkov in his proof† that $G^{3,7,7}$ and $G^{3,7,11}$ collapse, and in his proof‡ that

$$G^{3,7,9} \sim \mathfrak{P}_1(8). \tag{3.67}$$

(This is the simple group of order 504.)

Moreover, Sinkov has achieved a further simplification of the definition, in the case when n (as well as p) is odd; namely, he finds§ that the period of Q need not be specified. Thus $G^{3,n,p}$ (n, p odd) is defined by

$$P^n = (P^2Q)^3 = (P^3Q)^2 = (PQ^{(p+1)/2})^2 = 1. \tag{3.68}$$

For example, $\mathfrak{P}_1(8)$ is defined by

$$P^7 = (P^2Q)^3 = (P^3Q)^2 = (PQ^5)^2 = 1 \tag{3.69}$$

or, interchanging n and p, by $P^9=(P^2Q)^3=(P^3Q)^2=(PQ^4)^2=1$.

3.7. **Cases in which m, n, p are all odd:** $G^{3,5,5}$, $G^{3,7,9}$, $G^{3,7,13}$, $G^{3,7,15}$, $G^{3,9,9}$, $G^{5,5,5}$. In §3.2, we found $G^{3,5,5}$ to be the icosahedral group, as generated by

$$A = (2\ 5\ 3), \qquad B = (1\ 2\ 3\ 4\ 5), \qquad C = (1\ 2\ 4\ 3\ 5).$$

Clearly, A and B generate the same group in the form

$$A^3 = B^5 = (AB)^2 = 1,$$

while B and C generate it *qua* $(5, 5 \mid 2, 3)$.

We easily verify that $\mathfrak{P}_1(8)$, in the form $G^{3,7,9}$, is generated by the linear fractional substitutions

$$A = \left(\frac{1}{x+1}\right), \qquad B = (\alpha^2x + 1), \qquad C = \left(\frac{\alpha^5x+1}{\alpha^3x+\alpha}\right),$$

* Sinkov [3], p. 68.
† Sinkov [5].
‡ Sinkov [3], p. 70.
§ Sinkov [3], p. 69.

where α is a primitive root in the Galois field $GF(2^3)$, defined by

$$\alpha^3 + \alpha + 1 \equiv 0 \pmod{2}.$$

In the form (3.69), we therefore have

$$P = (\alpha^2 x + 1), \qquad Q = \left(\frac{\alpha}{\alpha^4 x + 1}\right).$$

We still more easily verify that the defining relations for $G^{3,7,13}$ are satisfied by the linear fractional substitutions

$$A = \left(\frac{1}{1-x}\right), \qquad B = \left(\frac{1}{x} - 1\right), \qquad C = (x+1) \pmod{13}.$$

Hence $\mathfrak{P}_1(13)$ occurs as a factor group. (In Theorem G, below, we shall generalize this result.) Sinkov has proved* that the order of $G^{3,7,13}$ is 2184, which is only twice that of $\mathfrak{P}_1(13)$. In fact, $\mathfrak{P}_1(13)$ is the quotient group of the central (of order two) generated by $(P^6Q^2)^3$, $(PQ^3)^3$, $(BC^6)^3$, or $(CBA)^7$.

Sinkov has remarked that $G^{3,7,13}$ does not contain $\mathfrak{P}_1(13)$ as a subgroup. For, such a subgroup, being of index two, would be invariant; and, having an abelian quotient group (of order two), it would contain the commutator subgroup of $G^{3,7,13}$; but $G^{3,7,13}$ is known to be *perfect*,† since it is generated by two operators (AB and A) of periods two and three, whose product is of period seven.

Thus $G^{3,7,13}$ is not isomorphic with $G^{3,7,12}$ or $G^{3,7,14}$, although it has the same order (2184). Nor is it the group of unimodular substitutions on two variables (mod 13),‡ since, if we represent A, B, C by the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -5 & 5 \\ 5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

we find that

$$(CA)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Sinkov has observed that the defining relations for $G^{3,7,15}$ are satisfied by the linear fractional substitutions

$$A = \left(\frac{4x+16}{x+26}\right), \qquad B = \left(\frac{22x+11}{12x+10}\right), \qquad C = \left(\frac{10x+1}{12x+10}\right) \pmod{29}.$$

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* Sinkov [3], p. 73.

† Brahana [2], p. 347.

‡ Dickson's $SLH(2, 13)$; van der Waerden's $SL(2, 13)$. That group is defined by

$$A^3 = B^7 = C^{13} = I^2 = 1, \quad (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = (CBA)^7 = I.$$

He and I have investigated this group, in the form

$$P^7 = (P^2Q)^3 = (P^3Q)^2 = (PQ^8)^2 = 1.$$

By enumerating the 406 cosets of the subgroup of order thirty generated by Q and PQP^2, we established the order as being 12180. Hence

$$G^{3,7,15} \sim \mathfrak{P}_1(29). \tag{3.71}$$

Similarly, the defining relations for $G^{3,9,9}$ are satisfied by the linear fractional substitutions

$$A = \left(\frac{6x+13}{2x+14}\right), \qquad B = \left(\frac{4x+7}{7x+3}\right), \qquad C = \left(\frac{2x}{10}\right) \pmod{19}.$$

By enumerating the 190 cosets of the subgroup of order eighteen generated by AB and C, J. M. Kingston* obtained the order 3420. Hence

$$G^{3,9,9} \sim \mathfrak{P}_1(19). \tag{3.72}$$

Brahana has shown† that $\mathfrak{P}_1(11)$ is generated by the permutations

$$S = (a\ g\ e\ b\ k)(c\ j\ i\ d\ f), \qquad T = (a\ c)(b\ d)(e\ g)(f\ h).$$

Writing

$$B = S^{-1} = (a\ k\ b\ e\ g)(j\ c\ f\ d\ i), \quad C = ST = (b\ k\ c\ j\ i)(f\ a\ e\ d\ h),$$

we observe that these permutations satisfy the relations

$$B^5 = C^5 = (BC)^2 = (B^2C^2)^5 = (B^3C^2B^2C^2)^2 = 1,$$

which, by (3.61), define $G^{5,5,5}$. By enumerating the 66 cosets of the subgroup of order ten generated by BC and CB, we easily find that these relations suffice to define a group of order 660. Hence‡

$$G^{5,5,5} \sim \mathfrak{P}_1(11). \tag{3.73}$$

The third generator for $G^{5,5,5}$ (in the form (3.11)) is

$$A = (B^2C^2)^2 = (S^{-1}TST)^2 = (c\ k\ a\ f\ h)(e\ b\ j\ d\ g).$$

A, B, C are cyclically permuted by $D^{-1} = (a, b, c)(e, j, f)(g, i, h)$. By adjoining this operator, we obtain the direct product $\mathfrak{P}_1(11) \times G_3$ in the form

---

* On July 14, 1938.

† Brahana [3], p. 546.

‡ Lewis [1], defines this group in the form $S^{11} = T^2 = (ST)^3 = (S^4TS^6T)^2 = 1$. Cf. Brahana [2], p. 356, where two other elegant definitions are suggested: $P^{11} = Q^5 = (PQ)^2 = (P^2Q^2)^2 = (P^8Q)^3 = 1$ $(P^2 = R)$, $R^{11} = T^2 = (RT)^3 = (R^{-1}TRT)^5 = (R^{-3}TR^3T)^2 = 1$. For another form of this last definition, see Todd and Coxeter [1], p. 32.

$$C^5 = D^3 = (CD)^6 = (CDCD^{-1})^2 = 1.*$$

$G^{4,4,4}$ leads analogously to the group

$$C^4 = D^3 = (CD)^6 = (CDCD^{-1})^2 = 1,$$

of order 192, which is generated by the permutations

$$(a_1\ b_1)(a_3\ b_3)(a_2\ b_2\ a_4\ b_4), \qquad (a_2\ b_2\ b_3)(a_4\ b_4\ b_1);$$

and $G^{2,2,2}$ ($G^{2,2,2} \sim G_2 \times G_2 \times G_2$) leads to the pyritohedral group (2, 3, 6; 2).

So too, the icosahedral group†

$$A^3 = B^3 = C^3 = (AB)^2 = (BC)^2 = (CA)^2 = 1 \tag{3.74}$$

leads to the group

$$C^3 = D^3 = (CDCD^{-1})^2 = 1, \tag{3.75}$$

of order 180. This is $G_{5!/2} \times G_3$, since the generators

$$A = (a_1\ a_4\ a_5), \qquad B = (a_2\ a_4\ a_5), \qquad C = (a_3\ a_4\ a_5),$$

of (3.74) are cyclically permuted by $(a_1\ a_2\ a_3)$. The relations (3.75) imply $(CD)^{15} = 1$; but $(a_3\ a_4\ a_5)$ and $(a_1\ a_2\ a_3)$ generate the factor group

$$C^3 = D^3 = (CD)^5 = (CDCD^{-1})^2 = 1,$$

which is therefore the icosahedral group again. Clearly, D and $(CD)^2$ satisfy the ordinary definition $D^3 = E^5 = (DE)^2 = 1$. More generally, the relations

$$C^k = D^3 = (CD)^5 = (CDCD^{-1})^l = 1$$

define $(l, 5 \mid 3, k)$. (The interesting cases are when $l = 3$ and $k = 3$ or 4.)

It is perhaps appropriate to remark here that the octahedral group $G^{3,3,4}$ appears naturally as a factor group of the (unextended) hyper-octahedral group‡ $A^3 = B^3 = C^4 = (AB)^2 = (BC)^2 = (CA)^2 = 1$, of order 192.

3.8. **The criterion for finiteness.** By applying §3.1 to the groups considered in Theorem C, we immediately deduce the following theorem:

THEOREM E. *If the smallest of m, n, p is greater than 2, while the next is greater than 3, and if these three numbers are either all even, or one even and the other two equal, the group $G^{m,n,p}$ is finite when*

$$\cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p < 1, \tag{3.81}$$

and infinite otherwise.

---

* In terms of CD and $DCDCD$, this takes the form $S^6 = T^2 = (S^2T)^3 = (S^3T)^5 = 1$. Hence yet another definition for $\mathfrak{P}_1(11)$ is $S^6 = T^2 = (ST)^{11} = (S^2T)^3 = (S^3T)^5 = 1$.

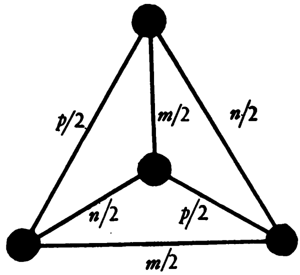
† Carmichael [1], p. 255.

‡ Coxeter [3], p. 219.

Since this result is formally more elegant than the analogous Theorems A and C, it seems worth while to give a direct proof.

When m, n, p are all even, we take the group generated by reflections

$$R_1^2 = R_2^2 = R_3^2 = R_4^2 = (R_1R_4)^{m/2} = (R_2R_3)^{m/2}$$
$$= (R_2R_4)^{n/2} = (R_3R_1)^{n/2} = (R_3R_4)^{p/2} = (R_1R_2)^{p/2} = 1,$$



and adjoin a four-group $M^2=N^2=P^2=MNP=1$, such that

$$\begin{aligned} R_1M &= MR_4, & R_2M &= MR_3 & &= A \text{ (say)},\\ R_2N &= NR_4, & R_3N &= NR_1 & &= B \text{ (say)},\\ R_3P &= PR_4, & R_1P &= PR_2 & &= C \text{ (say)}. \end{aligned}$$

These relations imply

$$M = BC, \qquad N = CA, \qquad P = AB,$$
$$R_1 = CAB, \qquad R_2 = ABC, \qquad R_3 = BCA, \qquad R_4 = CA^3B = AB^3C = BC^3A.$$

By direct substitution, the augmented group is seen to be $G^{m,n,p}$.

The necessary and sufficient condition for the group generated by reflections to be finite is*

$$\begin{vmatrix} 1 & -\cos 2\pi/p & -\cos 2\pi/n & -\cos 2\pi/m \\ -\cos 2\pi/p & 1 & -\cos 2\pi/m & -\cos 2\pi/n \\ -\cos 2\pi/n & -\cos 2\pi/m & 1 & -\cos 2\pi/p \\ -\cos 2\pi/m & -\cos 2\pi/n & -\cos 2\pi/p & 1 \end{vmatrix} > 0,$$

that is,

$$(1-\cos 2\pi/m-\cos 2\pi/n-\cos 2\pi/p)(1-\cos 2\pi/m+\cos 2\pi/n+\cos 2\pi/p)$$
$$\cdot(1+\cos 2\pi/m-\cos 2\pi/n+\cos 2\pi/p)(1+\cos 2\pi/m+\cos 2\pi/n-\cos 2\pi/p)>0.$$

---

* Strictly, we should mention the necessary condition $1/m+1/n+1/p>1/2$, which ensures that the solid angle at a vertex of the fundamental region has a spherical excess. But this condition is automatically covered by the other. (An analogous remark could have been made in §1.2.)

Since we are supposing m, n, p to be even and greater than 2, none of the cosines can be negative; so the last three factors are essentially positive, and may be discarded. (It will appear later, however, that they have a certain significance when we allow m, n, or p to be odd. See Theorem F.)

When $m=n$, and p is even, we observe that the subgroup $(2, m, m; p/2)$ is also a subgroup of index two in the group

$$S^m = R_0^2 = (R_0S^{-1}R_0S)^2 = (R_0S^{-2}R_0S^2)^{p/2} = 1,$$

whose order is m times that of the group generated by reflections

$$R_i^2 = (R_iR_{i+1})^2 = (R_iR_{i+2})^{p/2} = 1, \qquad R_{i+m} = R_i.$$

This is infinite when $m>5$, and again when $m=5$ and $p/2>2$. These values agree with the trigonometrical criterion. (When $m=3$, the groups collapse unless $p=4$.)

This completes the proof of Theorem E, which tells us, for instance, that $G^{4,6,6}$ and $G^{5,5,6}$ are infinite.

The groups $G^{4,4,2k}$ and $G^{4,5,5}$ are the only finite groups which satisfy all the conditions of Theorem E. If we relax the conditions of parity and equality, we find that the criterion (3.81) admits the groups:

$G^{3,4,p}$ (collapsing unless $p=3$ or 6),
$G^{3,5,p}$ (collapsing unless $p=5$ or 10),
$G^{3,6,p}$ (collapsing if p is odd),
$G^{3,7,p}$ ($p\leqq 12$; collapsing unless $p=8$, 9, or 12),
$G^{3,8,8}$, $G^{4,5,6}$, $G^{5,5,5}$,
$G^{4,4,p}$ (p odd), $G^{3,8,9}$, $G^{4,5,7}$ (collapsing),

all of which have already been described. Thus $G^{m,n,p}$ is finite (or collapses) whenever (3.81) is satisfied. Moreover,* with the exception of $G^{2,n,2n}$ (n odd), it collapses whenever

$$(1 - \cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m - \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m + \cos 2\pi/n - \cos 2\pi/p) < 0.$$

Hence we have the following theorem:

THEOREM F. *For all (noncollapsing) groups $G^{m,n,p}$, save $G^{2,n,2n}$ (n odd),*

$$(1 - \cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m - \cos 2\pi/n + \cos 2\pi/p)(1 + \cos 2\pi/m + \cos 2\pi/n - \cos 2\pi/p) \geqq 0;\dagger$$

and for all infinite groups $G^{m,n,p}$,

* See Figs. 3–5, below.

† The groups for which this expression vanishes, and the one exceptional group for which it is negative, are just the extended polyhedral groups $[m, n]$ (see (3.23)).

(3.82) $$\cos 2\pi/m + \cos 2\pi/n + \cos 2\pi/p \geqq 1.$$

Among the infinite groups, I would mention $G^{3,8,12}$ (see (2.67)). On the other hand, we have seen that the groups

$$G^{3,7,13},\quad G^{3,7,14},\quad G^{3,7,15},\quad G^{3,8,10},\quad G^{3,8,11},\quad G^{4,5,8},\quad G^{4,5,9},\quad G^{4,6,7}$$

are finite, although they satisfy (3.82).

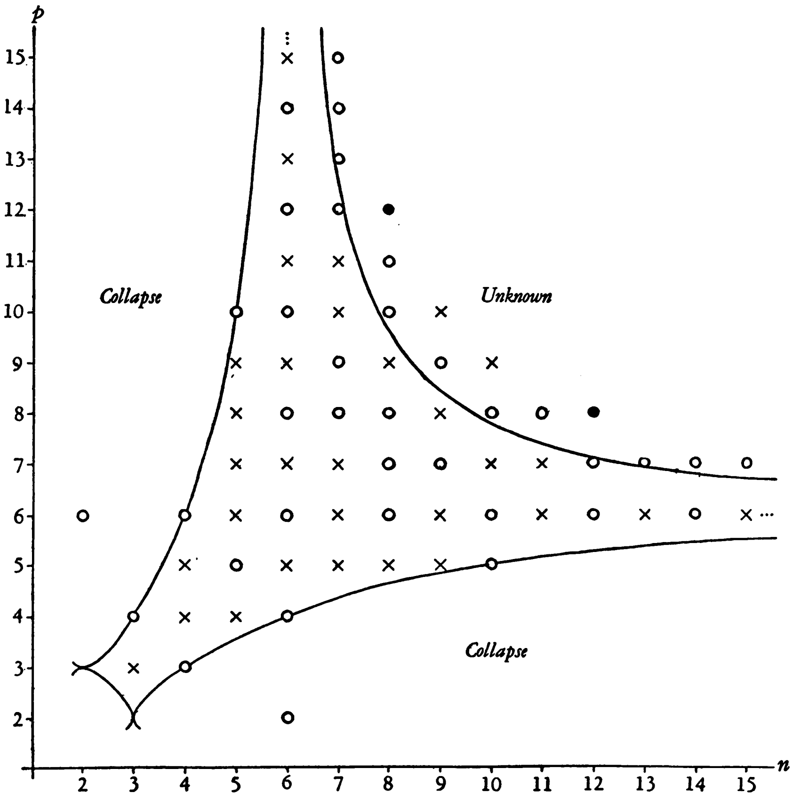


FIG. 3
A graphical enumeration of groups $G^{3,n,p}$

These results are collected in Table III, which shows that, with a few exceptions, the order of $G^{m,n,p}$ increases as

$$1 - \cos 2\pi/m - \cos 2\pi/n - \cos 2\pi/p$$

diminishes.

In Figs. 3, 4, 5,* we represent the particular groups $G^{3,n,p}$, $G^{4,n,p}$, $G^{5,n,p}$ (respectively) as points with Cartesian coordinates (n, p). The known finite groups are marked as o's, the known infinite groups as dots, and the cases of collapse as crosses (except where, n or p being small, collapse is general). The

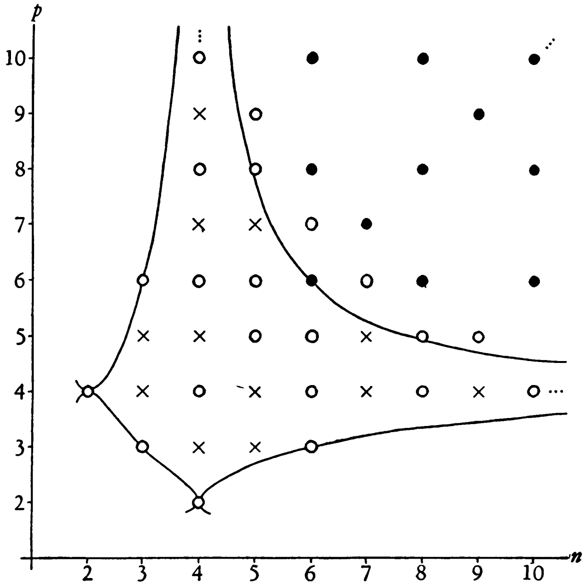


FIG. 4
A graphical enumeration of groups $G^{4,n,p}$

trigonometrical criteria appear as curves (which roughly enclose the region in which finite groups are prevalent). Clearly, all the known finite groups $G^{m,n,p}$ are included, some several times over.

3.9. $G^{3,n,p}$ **and the Fibonacci numbers.** Let f_n denote the nth Fibonacci number, that is, the nth term of the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, · · · , defined by the recurrence formula

$$f_1 = f_2 = 1, \qquad f_n + f_{n+1} = f_{n+2}.$$

Clearly,

$$f_n = 5^{-1/2}(\tau^n - \tau'^n),$$

where τ, τ' are the roots $(1 \pm 5^{1/2})/2$ of the equation

$$X^2 - X - 1 = 0.$$

* Cf. Coxeter [7], p. 41, where the groups $(l, m \mid 2, 3)$ and $(l, m \mid 2, 4)$ are similarly represented.

We shall see in a moment a connection between these numbers and certain groups $G^{3,n,p}$, where p is an odd prime.

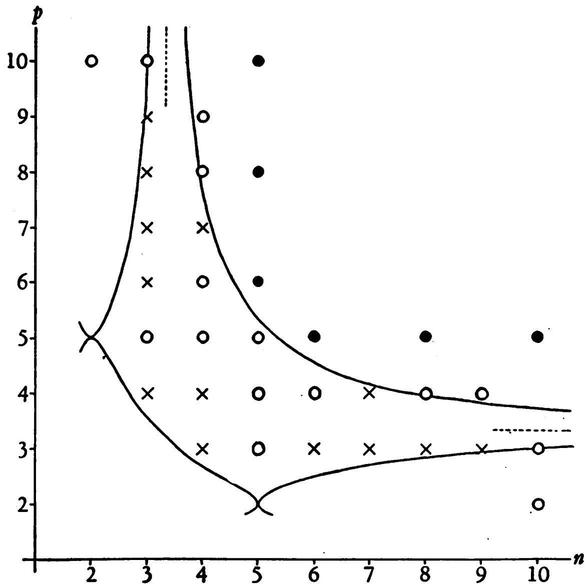


FIG. 5
A graphical enumeration of groups $G^{5,n,p}$

Consider the linear fractional substitutions

$$(3.91) \qquad A = \left(\frac{1}{1-x}\right), \qquad B = \left(\frac{1}{x} - 1\right), \qquad C = (x+1),$$

which are to be regarded as operating in the field of integers modulo p, an odd prime. It is easily verified that

$$AB = (-x), \qquad BC = \left(\frac{1}{x}\right), \qquad CA = \left(-\frac{1}{x}\right), \qquad ABC = (1-x).$$

Hence

$$A^3 = C^p = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1\,;$$

that is, all the defining relations for $G^{3,n,p}$ are satisfied, save that n, the period of B, remains to be determined. The matrix of B, namely

$$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

having the characteristic equation

$$X^2 + X - 1 = 0,$$

is similar to*

$$\begin{pmatrix} -\tau & 0 \\ 0 & -\tau' \end{pmatrix}.$$

We are thus led to consider the substitution

$$\left(\frac{\tau x}{\tau'}\right)$$

operating in the field of algebraic integers $a+b\tau$. We seek the smallest n for which

$$(\tau/\tau')^n \equiv 1 \pmod{p},$$

that is,

$$\tau^n \equiv \tau'^n \pmod{p},$$

$$5^{1/2} f_n \equiv 0 \pmod{p},$$

or (since f_n is a rational integer),

$$f_n \equiv 0 \pmod{p}.$$

Such a number n can always be found; in fact, by a known property of the Fibonacci numbers, n is a divisor of either $p+1$ or $p-1$ (except when $p=5$). With this definition for n, the given substitutions A, B, C satisfy all the defining relations for $G^{3,n,p}$.

Now, it is well known† that A and C (or CA and C) generate the linear fractional group $\mathfrak{P}_1(p)$. The remaining substitution B, having determinant -1, belongs to this group, or enlarges it to $\tilde{\mathfrak{P}}_1(p)$, according as -1 is a quadratic residue or nonresidue, that is, according as $p\equiv 1$ or $3 \pmod 4$. In the latter case, n is necessarily even; so A and C satisfy the defining relations for $(2, 3, p; n/2)$. Hence we have the following theorem:

THEOREM G. *If p is a prime, congruent to* 1 *or* 3 (mod 4), *the group* $\mathfrak{P}_1(p)$ *or* $\tilde{\mathfrak{P}}_1(p)$ *(respectively) is a factor group of* $G^{3,n,p}$, *where n is the ordinal of the first Fibonacci number that is divisible by p. When $p\equiv 3$* (mod 4), $\mathfrak{P}_1(p)$ *is a factor group of* $(2, 3, p; n/2)$.

Here are the first nine values of n:

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* For this method for calculating n, I am indebted to H. Hasse.

† Dickson [1], pp. 300–302.

p	3	5	7	11	13	17	19	23	29	$\cdots$
n	4	5	8	10	7	9	18	24	14	$\cdots$

Since A and C generate $\mathfrak{P}_1(p)$, B must be expressible in terms of A and C whenever $p\equiv 1 \pmod 4$. In fact, since

$$(a^2x) = \left(-\frac{1}{x}\right)(x+a)\left(-\frac{1}{x}\right)\left(x+\frac{1}{a}\right)\left(-\frac{1}{x}\right)(x+a)$$
$$= CA\cdot C^a\cdot CA\cdot C^{1/a}\cdot CA\cdot C^a = A^{-1}C^aAC^{1/a}A^{-1}C^{a-1},$$

we have, when $p\equiv 1 \pmod 4$,

$$AB = (-x) = A^{-1}C^iAC^{-i}A^{-1}C^{i-1},$$

where $i^2\equiv -1 \pmod p$. Thus

$$B = AC^iAC^{-i}A^{-1}C^{i-1}. \tag{3.92}$$

In Todd's definition* for $\mathfrak{P}_1(p)$,

$$S = C, \qquad U = A, \qquad R = A^{-1}C^aAC^{1/a}A^{-1}C^a.$$

When $p=3$, 5, or 7, $\mathfrak{P}_1(p)$ or $\tilde{\mathfrak{P}}_1(p)$ is the *whole* group $G^{3,n,p}$ (see §3.2, §3.3). But, as we saw in §3.7,

$$\mathfrak{P}_1(13) \sim G^{3,7,13}/G_2. \tag{3.93}$$

It seems unlikely that $G^{3,n,p}$ is finite in any higher case.

We observe that n is even when $p=29$. In this case, then, (3.92) (with $i=12$) cannot be a consequence of the defining relations for $G^{3,n,p}$, since each of those relations involves B an even number of times. But I do not believe that this extra relation will suffice to reduce $G^{3,14,29}$ to a finite group.

Chapter IV. Graphical representation

4.1. **Dyck's general group picture.** Dyck† represents the group

$$S_1^{b_1} = S_2^{b_2} = \cdots = S_r^{b_r} = S_1S_2\cdots S_r = 1 \tag{4.11}$$

by a "group picture" (*Dycksche Gruppenbild*) which consists of a network of r-gons, each of angles

$$\pi/b_1,\ \pi/b_2,\ \cdots,\ \pi/b_r,$$

filling the sphere, the euclidean plane, or the hyperbolic plane, according as

* Todd [2], p. 195. The period of R is, of course, $(p-1)/2$.

† Dyck [1], p. 26; Burnside [1], pp. 372–427; Threlfall [1], p. 26.

$$\frac{1}{b_1}+\frac{1}{b_2}+\cdots+\frac{1}{b_r}$$

is greater than, equal to, or less than, $r-2$. The polygons are supposed to be colored alternately white and black, and each side separates two polygons (of opposite colors) which are images of one another by reflection in that side. The group is transitive on the polygons of either color, and by adjoining one of the reflections we derive the *extended* group

$$(4.12)\qquad \begin{cases} R_1^2 = R_2^2 = \cdots = R_r^2 = 1, \\ (R_rR_1)^{b_1} = (R_1R_2)^{b_2} = \cdots = (R_{r-1}R_r)^{b_r} = 1, \end{cases}$$

for which the polygon is the fundamental region. The generators $R_1, R_2, \cdots, R_r$ are reflections in the sides of the fundamental region. The S's are their products in consecutive pairs. The groups are finite when the representation is on a sphere, in which case $r=2$ or 3 and (when $r=3$) we have the relation $1/b_1+1/b_2+1/b_3>1$.

For many purposes, it is desirable to abandon the metric, and to regard the group picture as a network of polygons on a topological surface. We can then derive a group picture for any factor group, given by the insertion of extra relations in the abstract definition, by making the appropriate identification of various parts of the surface. (Such identification naturally reduces a simply connected surface to a multiply connected one.)

To obtain a group picture for the group derived from (4.12) by adding the relation

$$(4.121)\qquad (R_1R_2\cdots R_r)^p = 1,$$

we should identify, in the group picture for (4.12) itself (that is, the "universal covering surface") all those polygons which represent operators of the subgroup generated by $(R_1R_2\cdots R_r)^p$ and its conjugates. If rp is odd, this will involve the identification of a white region with a black one, so that the new surface will be unorientable. (Dyck would have represented the same group on the orientable twofold covering surface.)

4.2. **The group picture for $G^{m,n,p}$.** In the case when $r=3$ and b_1, b_2, b_3 are $2, m, n$, we have a factor group of $[m, n]$, namely

$$R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^m = (R_2R_3)^n = (R_3R_1)^2 = (R_1R_2R_3)^p = 1.$$

We saw, in §3.2, that this is precisely $G^{m,n,p}$, the ordinary definition being given by the substitution

$$(4.21)\qquad \begin{aligned} R_1 &= BC, & R_2 &= BCA, & R_3 &= CA, \\ A &= R_1R_2, & B &= R_2R_3, & C &= R_3R_2R_1. \end{aligned}$$

When p is even, our group picture for $G^{m,n,p}$ is Dyck's for the subgroup $(2, m, n; p/2)$, which is generated by A, B (and AB). When p is odd, A and B generate the whole group $G^{m,n,p}$, and the distinction between white and black evaporates.

Klein* shows a stereographic projection of the group picture for [3, 5]. The sides of the triangles appear as arcs of circles, meeting at the proper

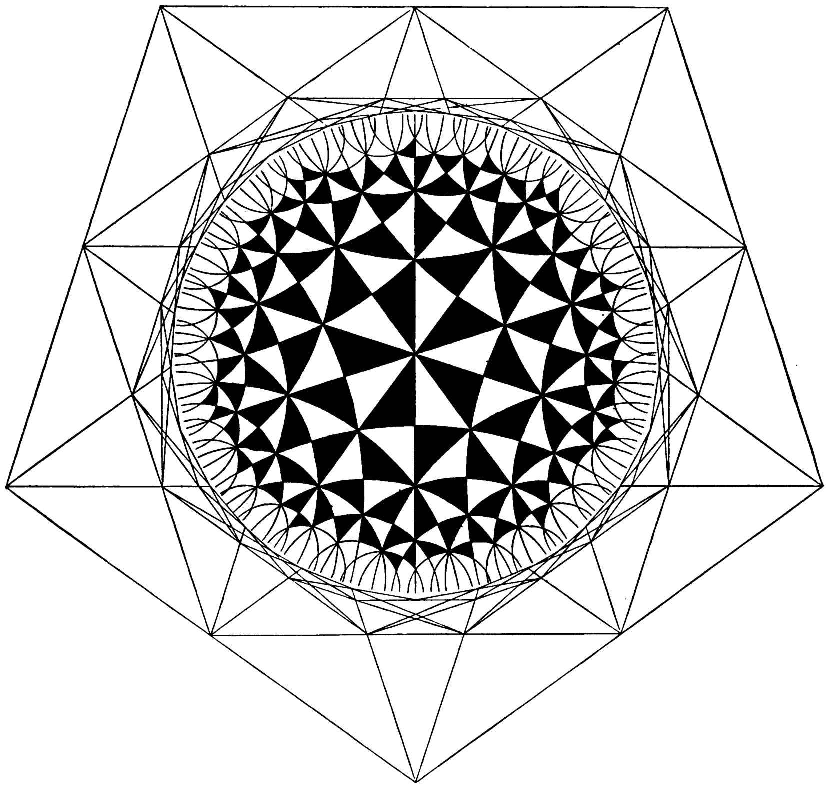

FIG. 6
The group picture for [4, 5]

angles, and the reflections are represented by inversions in these circles. Elsewhere† he and Fricke show the analogous conformal representation of the (hyperbolic) group picture for [3, 7]. For [4, 5] and [4, 6], see Figs. 6 and 7, where I have shown also the construction lines for the centers of the circles.

* Klein [1], p. 260.
† Klein-Fricke [1], p. 109, Fig. 33.

(Since the circles belong in various ways to coaxial systems, their centers lie conveniently on straight lines. In the case of [4, 6], all the centers are derivable from the outermost hexagon in a remarkably simple manner.)

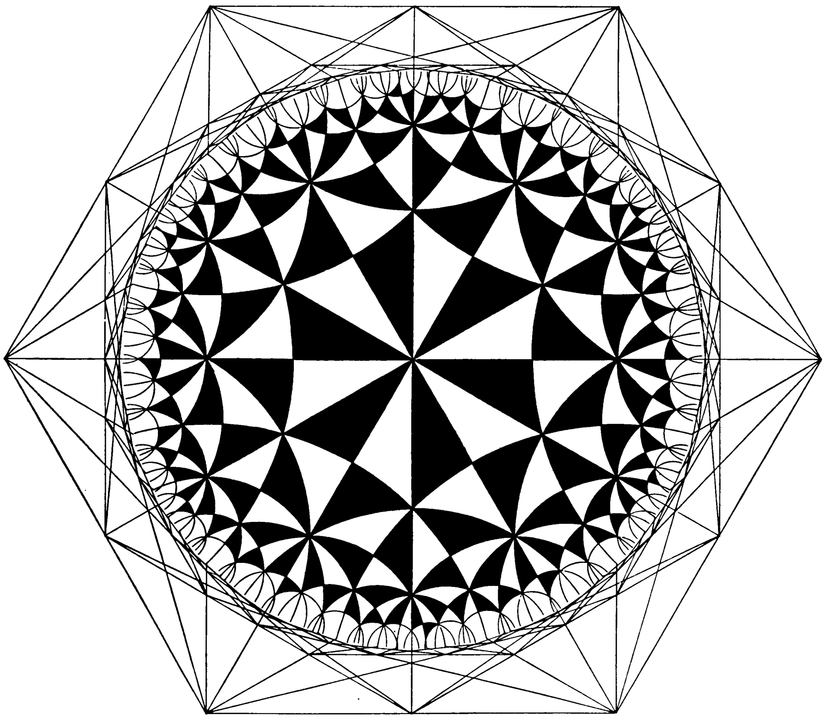

Fig. 7
The group picture for [4, 6]

4.3. **The regular map** $\{m, n\}_p$. Clearly, the vertices fall into three categories, say P_2, P_m, P_n, according as the number of triangles of either color that surround the vertex is 2, m, or n. The $2m$ triangles that surround one point P_m form together a regular m-gon, $\{m\}$, and the totality of such m-gons constitutes a *regular map** $\{m, n\}$, whose vertices (each surrounded by n $\{m\}$'s) are the points P_n. Similarly, the points P_m are the vertices of the reciprocal map $\{n, m\}$. The points P_2 are the mid-points of the edges of either map. In the case when $m=n$, the points P_m and P_n are surrounded alike; taken all together, they are the vertices of a regular map of quadrangles, $\{4, n\}$.

* Brahana [1]. This is the *regelmässige Zellsystem* $\{n, m\}$ of Threlfall [1], p. 32. Following Schläfli, van Oss, and Sommerville, I write the m (or a_2) before the n (or a_0).

Let P, Q, R, S, T, $\cdots$ be a sequence of vertices of $\{m, n\}$, so chosen that P, Q, R are consecutive vertices of one m-gon, Q, R, S of another, R, S, T of another, and so on. The edges PQ, QR, RS, $\cdots$ form a kind of zig-zag, which we call a *Petrie polygon.* (See Fig. 8, for the case of $\{5, 4\}$.) We easily see that the operator $R_1R_2R_3$ cyclically permutes the vertices (and sides) of a Petrie polygon. Consequently, the insertion of the relation $(R_1R_2R_3)^p=1$ corresponds to the identification of all pairs of vertices which are separated by p consecutive sides of a Petrie polygon. The reduced map, so derived, will be denoted by $\{m, n\}_p$. The map $\{5, 10\}_3$ is one of Brahana's and Coble's unorientable dodecahedra.*

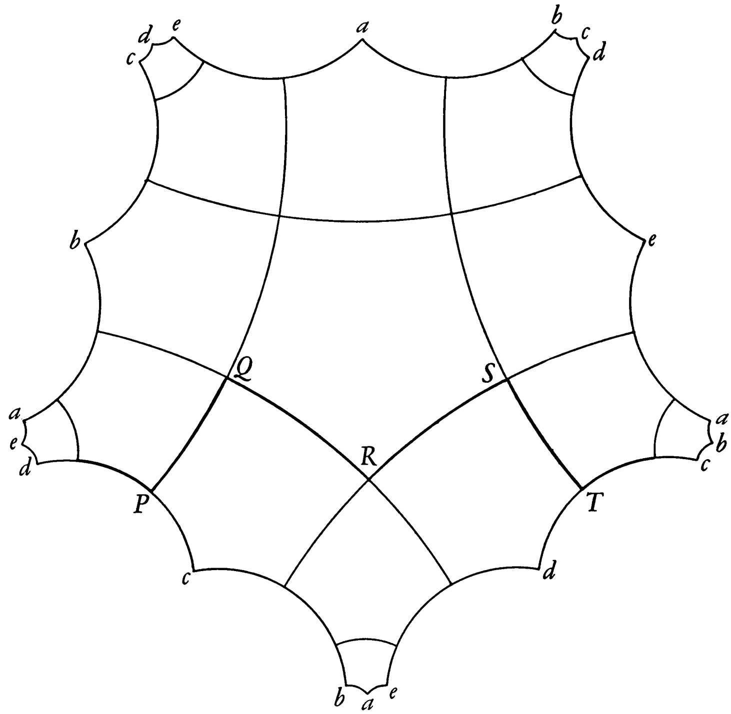

FIG. 8
The regular map $\{5, 4\}_5$

4.4. **The semi-regular map** $\{m/n\}_p$. In the metrical representation of $[m, n]$, the mid-points of the sides of the Petrie polygon lie on a straight line (of the euclidean or non-euclidean plane) or a circle (of the stereographic projection). This suggests the desirability of replacing the regular map by a *semi-regular map,* whose vertices are all the points P_2.

* Brahana and Coble [1], p. 15, Fig. VIII.

This semi-regular map, which we denote by $\{m/n\}$ (or $\{n/m\}$), has faces of two types, $\{m\}$ and $\{n\}$, whose centers are the points P_m and P_n, respectively. (When $m\neq n$, these faces are actually different; when $m=n$, they can still be distinguished, like the black and white squares of a checkerboard. See Fig. 11.) Every $\{m\}$ is surrounded by m $\{n\}$'s, every $\{n\}$ by n $\{m\}$'s, and every vertex by two of each, alternating. Moreover, when an edge is produced in both directions, the line obtained (corresponding to the Petrie polygon of $\{m, n\}$) is entirely covered with edges.

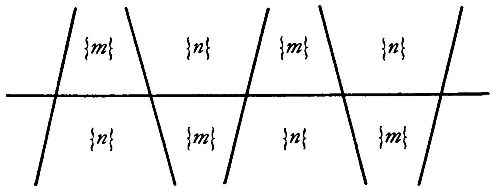

FIG. 9
Collinear edges of $\{m/n\}$

The operator $R_1R_2R_3$ permutes the edges by one step along such a line, and the relation $(R_1R_2R_3)^p=1$ corresponds to the identification of all pairs of vertices which differ by p such steps. When this identification has been effected, we denote the semi-regular map by* $\{m/n\}_p$, reserving the symbol $\{m/n\}$ itself for the universal covering map (which is $\{m/n\}_\infty$ unless $1/m+1/n>1/2$).

As we have already remarked, the map is unorientable when p is odd. In fact, since the collinear edges have alternate $\{m\}$'s and $\{n\}$'s on the left, and vice versa on the right, the surface in the neighborhood of the line forms a "Möbius band" (see Fig. 9).

When $m=n$, the map $\{m/n\}_p$ is regular instead of semi-regular. In particular, $\{m/m\}$ is $\{m, 4\}$, and $\{m/m\}_{2q}$ is the "skew polyhedron"† $\{m, 4|2q\}$, whose rotation group is $(m, 4\,|\,2, 2q)$ (see Fig. 18). We must *not* infer, however, that

$$G^{m,m,2q} \sim (m, 4 \mid 2, 2q),$$

although this happens to be true when $m=3$ or 5 and $q=2$ (see (3.31)). For the map represents the two groups in quite different ways: $(m, 4\,|\,2, 2q)$ contains no reflections, but $G^{m,m,2q}$ (containing reflections) will not transform an m-gon into any adjacent m-gon. All that we may infer is that the two groups

* Since $\{m/n\}_p$ has the same meaning as $\{n/m\}_p$, perhaps a better symbol would have been $\left\{ {m \atop n} \right\}_p$. The symbol $\left\{ {m \atop n} \right\}$, here written $\{m/n\}$, was defined in Coxeter [1], p. 127.

† Coxeter [7], p. 50.

have a common subgroup of index two, as we already know from (2.51) and §3.1, the subgroup being $(2, m, m; q)$.

Since $G^{m,n,p}$ involves m, n, p symmetrically, the three maps $\{m/n\}_p$, $\{n/p\}_m$, $\{p/m\}_n$ all represent the same group. (If $m \leqq n \leqq p$, the first of the three is usually the most convenient, as having the lowest connectivity.) The group $G^{m,n,p}$ is also represented by any of the six corresponding regular maps

$$\{m, n\}_p, \qquad \{n, m\}_p, \qquad \{n, p\}_m, \qquad \{p, n\}_m, \qquad \{p, m\}_n, \qquad \{m, p\}_n.$$

The map $\{m, 4\}_m$ has four $\{m\}$'s at each vertex, just like $\{m/m\}_4$; it represents the same group $G^{4,m,m}$, but has half as many $\{m\}$'s altogether. In fact, $\{m/m\}_4$ is its twofold covering surface. (See Figs. 8 and 18. The outermost edges are supposed to be identified in accordance with the lettering.)

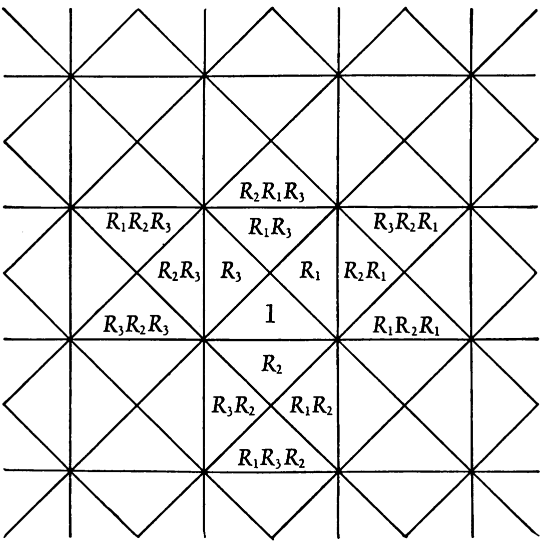

FIG. 10
$G^{4,4,4}$ as a factor group of $[4, 4]$

4.5. **The manner in which $\{m/n\}_p$ represents $G^{m,n,p}$.** Since each edge of $\{m/n\}_p$ crosses the common hypotenuse of two of Dyck's triangles (one white and one black), it is the half-edges that represent the operators of $G^{m,n,p}$. When p is even, the edges themselves represent the operators of the subgroup $(2, m, n; p/2)$.* Hence, if g denotes the order of $G^{m,n,p}$, the semi-

* When both n and p are even, the edges of $\{m/n\}_p$ can be indexed so as to form a *Cayley color-group* (Burnside [1], pp. 423–427) or *Dehnsche Gruppenbild* (Threlfall [1], pp. 22–27) for $(m, m \mid n/2, p/2)$, thus providing a geometrical interpretation for (2.52).

regular map $\{m/n\}_p$ has $g/2$ edges, $g/4$ vertices, $g/2m$ $\{m\}$'s, and $g/2n$ $\{n\}$'s. Its Euler-Poincaré characteristic* $(-V+E-F)$ is therefore

$$\frac{g}{2}\left(\frac{1}{2}-\frac{1}{m}-\frac{1}{n}\right).$$

The generators A, B, C are to be interpreted as follows. A is an m-gonal rotation about the center of a face $\{m\}$; B is an n-gonal rotation (in the same sense) about the center of an adjacent face $\{n\}$; and C, being the inverse of

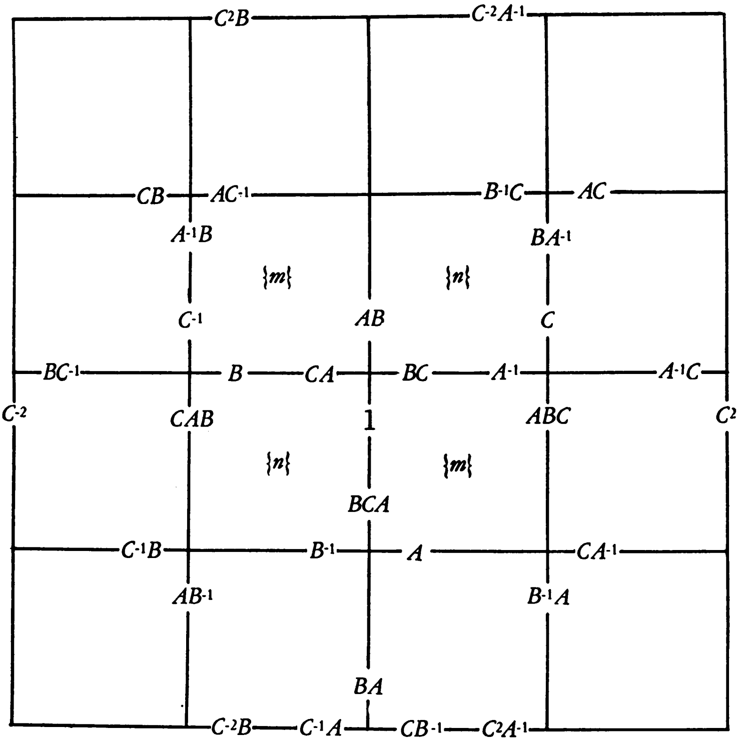

FIG. 11
$\{4/4\}_4$ as a portion of $\{4/4\}$

$R_1 R_2 R_3$, is a "glide" along a chain of collinear edges. To be precise, C transforms the edge one half of which represents B into that whose "other" half represents A^{-1}.

4.6. **Special cases.** Figs. 10 and 11 show the two kinds of group picture for $G^{4,4,4}$ (of order 64); namely, Dyck's picture for (2, 4, 4; 2), and the map

* An *orientable* surface of characteristic c has genus $c/2+1$.

$\{4/4\}_4$. Some of the operators have been explicitly inserted; the rest are easily deduced. The generalization to $G^{4,4,p}$ for any even p is obvious, as is the collapse of $G^{4,4,p}$ for any odd p.

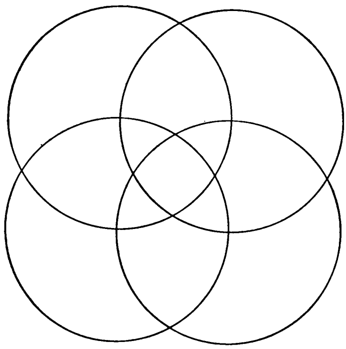

FIG. 12. $\{3/4\}_6$

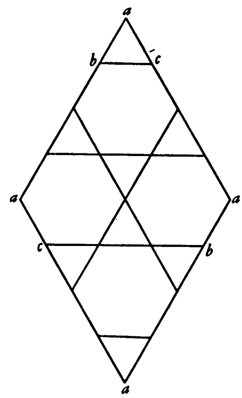

FIG. 13. $\{3/6\}_4$

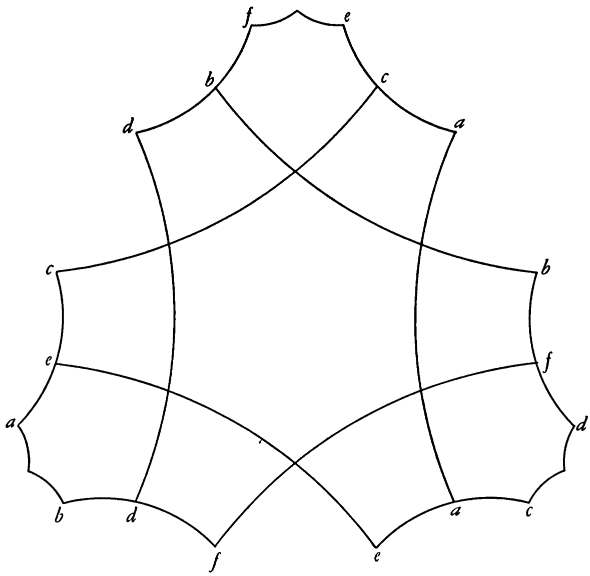

FIG. 14. $\{4/6\}_3$

The extended octahedral group $G^{3,4,6}$ provides an interesting example, as its three pictures (Figs. 12, 13, 14) are, respectively, spherical, euclidean, and hyperbolic. The map $\{3/6\}_4$ generalizes at once to $\{3/6\}_p$ for any even p. This can always be drawn as a rhombus (of angle $\pi/3$), with opposite sides identified, like the period parallelogram of an elliptic function.* By trying to make an analogous picture when p is odd, we verify the collapse of $G^{3,6,p}$ (p odd).

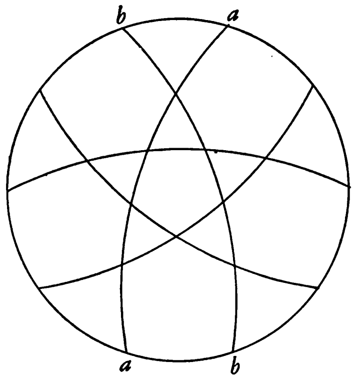

FIG. 15. $\{3/5\}_5$

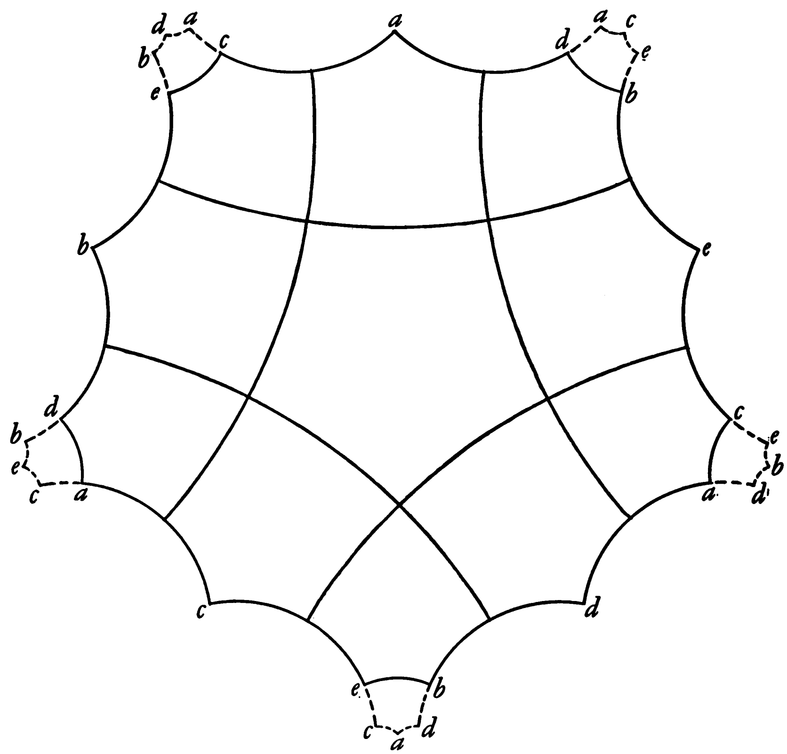

FIG. 16. $\{5/5\}_3$

* Edington [1], p. 206.

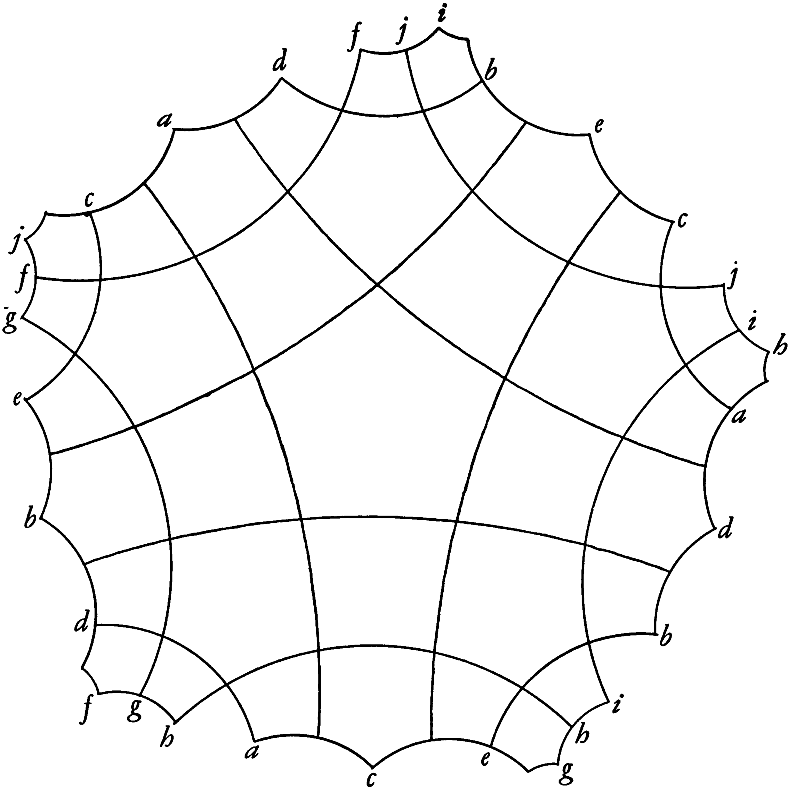

FIG. 17. $\{4/5\}_5$

Figs. 15 and 16 represent the icosahedral group $G^{3,5,5}$. In the former, all pairs of opposite points of the outermost circle are to be identified. If all six circles were completed, we should have $\{3/5\}_{10}$ or $\{3/5\}$. The map $\{3/5\}_5$ may be regarded as a partition of the elliptic plane into ten triangles and six pentagons; $\{3/5\}$ is its twofold covering surface. Similarly, $\{3/4\}$ is the twofold covering surface of $\{3/4\}_3$, which may be regarded as a partition of the elliptic plane into four triangles and three squares.

In Fig. 16, the smallest pentagon has been drawn five times over (in broken lines), to preserve the pentagonal symmetry of the figure. (The re-

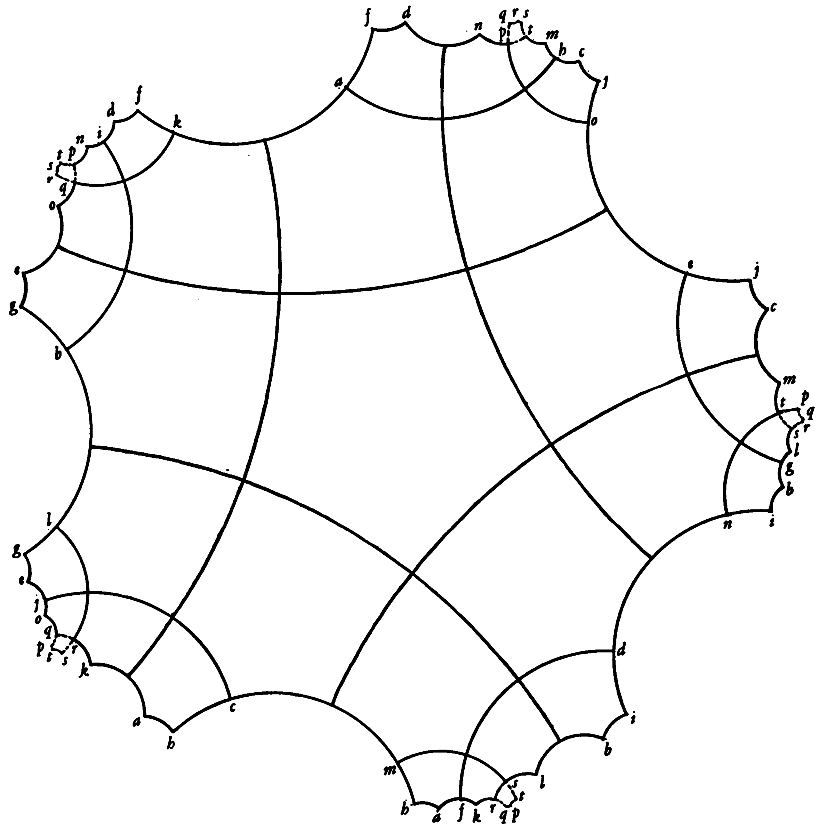

FIG. 18
$\{5/5\}_4$ or $\{5, 4 \mid 4\}$

semblance to Fig. 8 is therefore spurious.) $\{5/5\}_3$ is another of Brahana's and Coble's unorientable dodecahedra.* Its twofold covering surface is $\{5, 4\}_6$.

Figs. 17 and 18 represent the group $G^{4,5,5}$, of order 160. These are unorientable and orientable, respectively. The latter is the twofold covering sur-

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* Brahana and Coble [1], p. 3 (Fig. II). Similarly, $\{5, 6\}_4$ is the twofold covering surface of Fig. III (p. 9).

face of the unorientable map shown in Fig. 8. It represents also the subgroup (2, 5, 5; 2), of order 80, which is the abelian group of order sixteen and type (1, 1, 1, 1) augmented by an operator of period five.

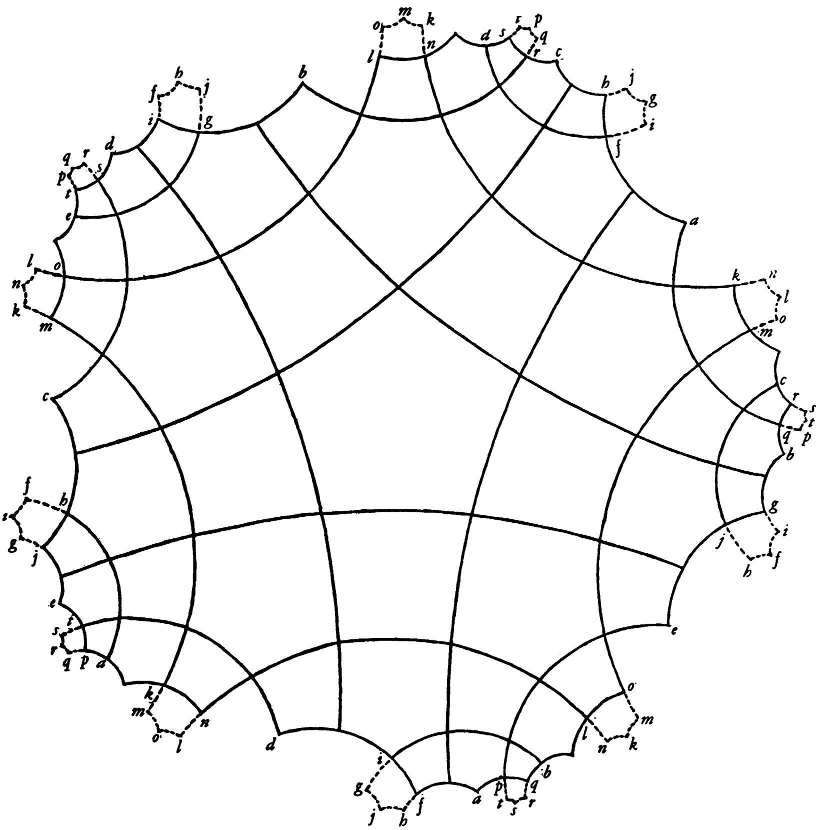

Fig. 19. $\{4/5\}_6$

Figs. 19 and 20 show two of the semi-regular maps for $G^{4,5,6}$. The former, being orientable, represents also the subgroup (2, 4, 5; 3), which is the symmetric group of degree five. (In Fig. 20, the small hexagon *mnopqr* has not been repeated, although it upsets the symmetry of the figure.)

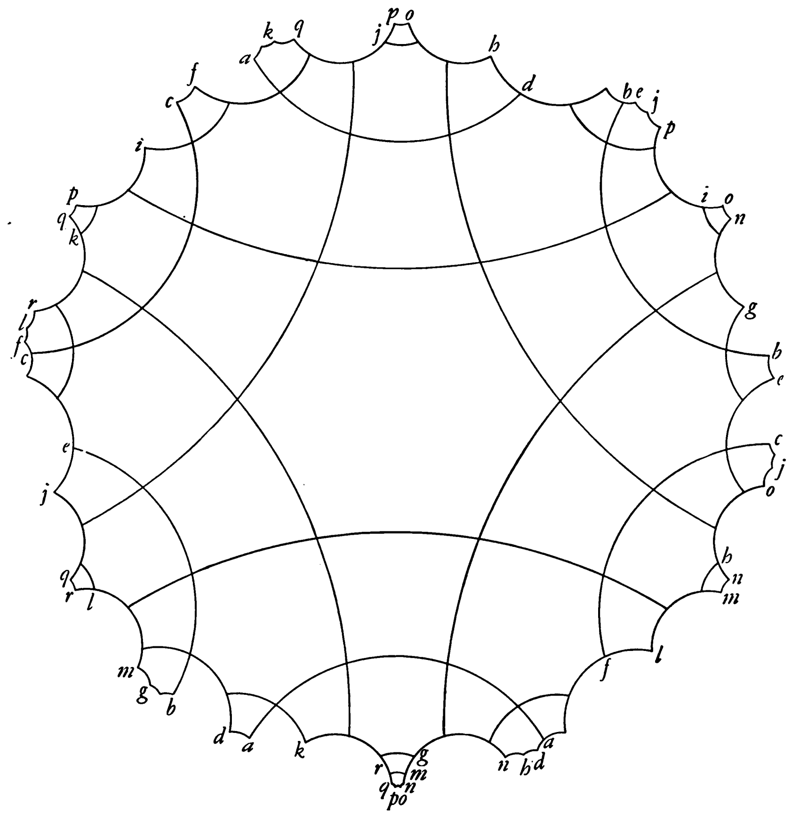

FIG. 20. $\{4/6\}_5$

Fig. 21 represents $G^{3,7,8}$, the 168 edges representing the operators of the subgroup (2, 3, 7; 4), which is $\mathfrak{P}_1(7)$.

Finally, Fig. 22 represents $G^{3,7,9}$, which is $\mathfrak{P}_1(8)$, the simple group of order 504. The peculiar elegance of this figure is partly due to the fact that the number of heptagons, being the number of Sylow subgroups of order seven, is one more than a multiple of seven, so that the heptagonal symmetry

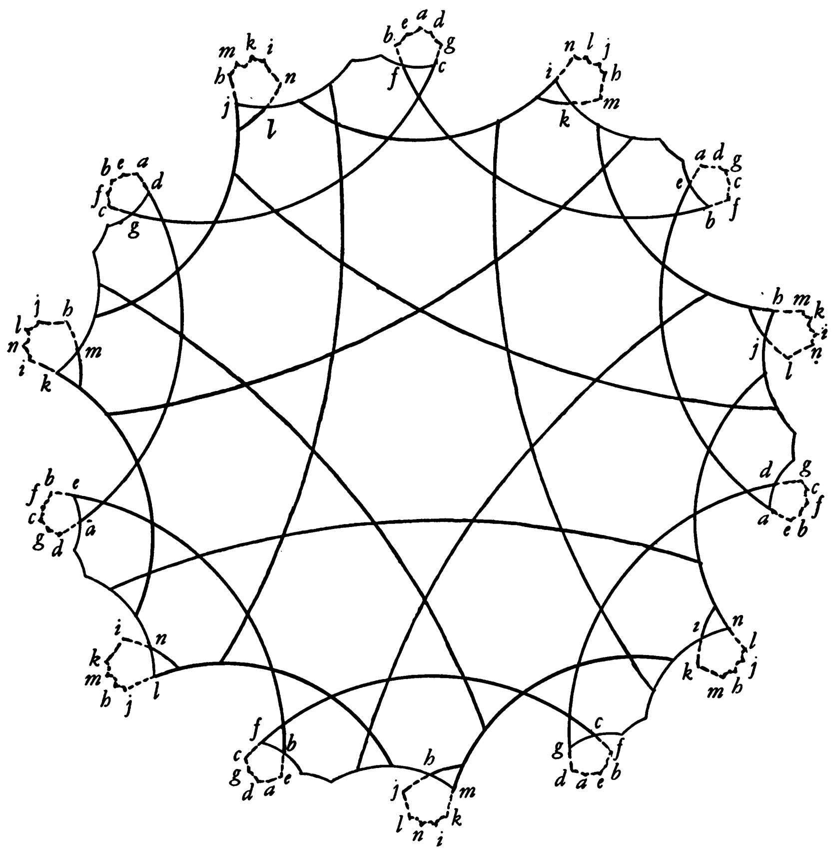

Fig. 21. $\{3/7\}_8$

can be preserved without repeating any of the faces. (In Fig. 21, there are *three* heptagons for every such subgroup of $G^{3,7,8}$.) Fig. 22 incidentally provides a geometrical verification for the collapse* of $G^{3,7,7}$. The numbers

* Cf. Sinkov [5].

1, 2, 3, 4, 5 have been inserted in four of the heptagons, each consecutive pair of which differ by seven steps along a line of collinear edges. To make

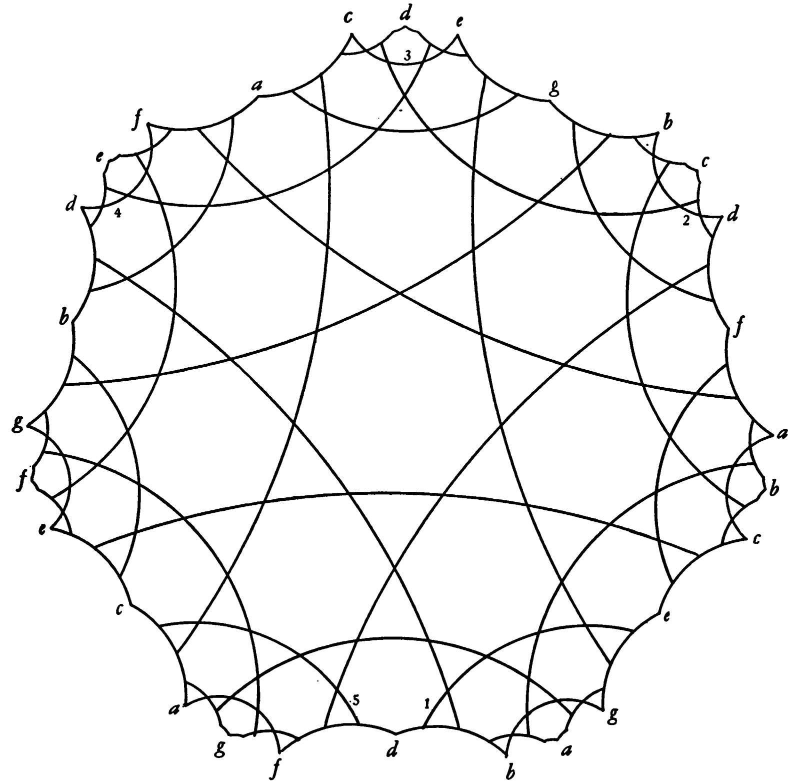

FIG. 22. $\{3/7\}_9$

$\{3/7\}_7$, these heptagons would have to be identified. But the numbers have been put against particular sides of the heptagons, namely, sides which would have to be identified. We are thus led to the identification of two sides (1 and 5) of one heptagon, which is absurd.

4.7. **The polyhedral groups.** When $1/m+1/n>1/2$, the vertices of the maps

$$\{m, n\}, \qquad \{n, m\}, \qquad \{m/n\},$$

regarded as points of a sphere, can be joined by straight lines and planes (in ordinary space) to form regular and semi-regular polyhedra. From this point of view,* $\{3, 3\}$ is the tetrahedron, $\{3/3\}$ or $\{3, 4\}$ is the octahedron, $\{4, 3\}$ is the cube, $\{3/4\}$ is the cuboctahedron,† $\{3, 5\}$ is the icosahedron, $\{5, 3\}$ is the dodecahedron, and $\{3/5\}$ is the icosidodecahedron.‡

When $\{m/n\}$ is regarded as a polyhedron, the "collinear edges" form an equatorial§ p-gon, C being a rotary reflection whose period p we proceed to calculate. Consider the four vertices which are joined by edges to any one vertex. These form a rectangle, whose sides are $2 \cos \pi/m$ and $2 \cos \pi/n$, while its diagonal is $2 \cos \pi/p$. Hence

$$\cos^2 \pi/p = \cos^2 \pi/m + \cos^2 \pi/n, \qquad 1/m + 1/n \geqq 1/2,$$

in agreement with (3.23).

This formula fails when it leads to an odd value for p, since then C^p, instead of being identity, is the reflection in the plane of the p-gon. This happens only when $m=2$ and n is odd (or *vice versa*). In order to cover this case, we may say that the period of C is twice the numerator of the rational number q, defined by

$$(4.71) \qquad \cos \pi/q = 1 + \cos 2\pi/m + \cos 2\pi/n.$$

4.8. **Polyhedra of higher connectivity.** From the ordinary polyhedron $\{m/n\}$ or $\{m/n\}_p$, we may derive two *star polyhedra*, $\{n/p\}_m$ and $\{m/p\}_n$, by regarding the equatorial p-gons as faces, and discarding the m-gons or n-gons, respectively. Thus we have the *tetratrihedron*|| $\{3/4\}_3$, the *hexatetrahedron*¶ $\{4/6\}_3$, the *octatetrahedron*¶ $\{3/6\}_4$, the *dodecahexahedron*** $\{5/10\}_3$, and the *icosihexahedron*** $\{3/10\}_5$.

The cuboctahedron, $\{3/4\}$ or $\{3/4\}_6$, represents the extended octahedral group $G^{3,4,6}$, which has two distinct subgroups of index two: (2, 3, 4; 3) and (2, 3, 6; 2). The former appears geometrically as the octahedral group, generated by a trigonal rotation A and a tetragonal rotation B, while the latter

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* Cf. Coxeter [1], p. 129.

† Badoureau [1], p. 67, Fig. 30; p. 73, Fig. 39.

‡ Ibid., p. 73, Fig. 45; p. 129, Figs. 113, 114.

§ That is, inscribed in a great circle.

|| The *semi-octaèdre* of Badoureau [1], p. 104, Fig. 70.

¶ Ibid., p. 119, Figs. 96, 97.

** Ibid., p. 131, Figs. 115, 116.

appears as the pyritohedral group, generated by the same A and a rotary reflection C. When the same group is represented on the octatetrahedron $\{3/6\}_4$ (whose faces are the eight triangles of the cuboctahedron, and its four equatorial hexagons), the roles of the two subgroups are interchanged: the generators, A and C, of the pyritohedral subgroup (2, 3, 6; 2), though still a rotation and a rotary reflection in space, are both *intrinsic* rotations* of the surface of the polyhedron. On the other hand, the third polyhedron, $\{4/6\}_3$ (whose faces are the six squares of $\{3/4\}_6$ and the four hexagons of $\{3/6\}_4$) is unorientable, and its intrinsic rotations, B and C, generate the whole (extended octahedral) group. Similarly, the intrinsic rotations of $\{3/4\}_8$ generate the extended tetrahedral group $G^{3,3,4}$ (which is simply isomorphic with the octahedral group), while those of $\{5/10\}_3$, or of $\{3/10\}_5$, generate the extended icosahedral group $G^{3,5,10}$.

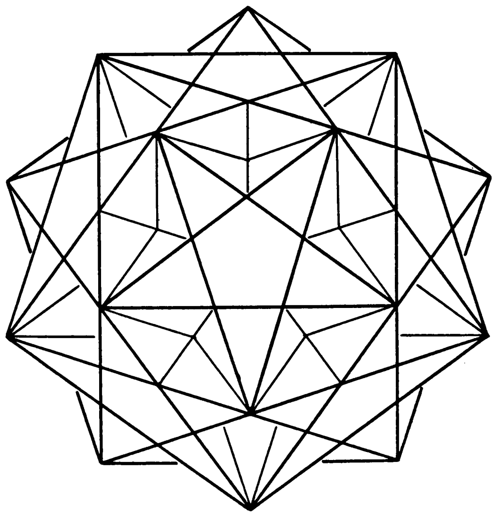

FIG. 23
The ditrigonal dodecadodecahedron, homeomorphic to $\{5, 6\}_4$

The cuboctahedron may be said to be *semi-reciprocal* to the cube and the octahedron. Semi-reciprocal to the Kepler-Poinsot polyhedra $\{5, \frac{5}{2}\}$ and $\{\frac{5}{2}, 5\}$ there is the dodecadodecahedron† $\{5/\frac{5}{2}\}$, which has the same vertices as the icosidodecahedron, but its faces consist of twelve pentagons and twelve pentagrams (or star pentagons). By ignoring the distinction between

* Cf. Coxeter [7], p. 47.
† Badoureau [1], p. 133, Fig. 117.

these two kinds of face (which, topologically speaking, are both pentagons), we derive the map $\{5, 4\}_6$, whose intrinsic rotation group, (2, 5, 4; 3), is $G_{5!}$. In fact, the pentagons and pentagrams of $\{5 / \frac{5}{2}\}$ are interchanged by any outer automorphism of the icosahedral group (which is the rotation group of the solid $\{5 / \frac{5}{2}\}$).

All this is closely analogous to the manner in which two reciprocal tetrahedra $\{3, 3\}$ lead to the octahedron (or "tetratetrahedron") $\{3/3\}$, which is the same polyhedron as $\{3, 4\}_6$. For, the octahedral group (2, 3, 4; 3) is the group of isomorphisms of the tetrahedral group.

Besides the dodecadodecahedron $\{5 / \frac{5}{2}\}$, there is another polyhedron (Fig. 23) having the same number of the same kinds of face (namely, pentagons and pentagrams), but three of each at a vertex, instead of only two. We may call this the *ditrigonal dodecadodecahedron*.* It has the same vertices as the ordinary dodecahedron. By ignoring the distinction between pentagons and pentagrams, we derive the map $\{5, 6\}_4$, whose intrinsic rotation group, (2, 5, 6; 2), is again $G_{5!}$.

4.9. **The more general group** $((b_1, b_2, \cdots, b_r; p))$. By analogy with (3.15), we use the symbol $((b_1, b_2, \cdots, b_r; p))$ to denote the group defined by (4.12) and (4.121). The substitution (4.21) enables us to define $((l, m, n; p))$ in the alternative form

$$(4.91) \qquad A^m = B^n = C^p = (AB)^l = (BC)^2 = (CA)^2 = (ABC)^2 = 1,$$

which shows that

$$(4.92) \qquad ((2, m, n; p)) \sim G^{m,n,p}.$$

Three methods present themselves for a tentative investigation of the group $((b_1, b_2, \cdots, b_r; p))$. The first applies to the case when $r=3$ and p is even. We saw, in §3.1, that $((l, m, n; 2q))$ *contains* $(l, m, n; q)$ *as a subgroup of index two*. By writing

$$R = R_3R_1, \qquad S = R_1R_2, \qquad T = R_2R_3,$$

so that

$$(R_1R_2R_3)^2 = SRT,$$

we obtain the subgroup in the symmetrical form (2.111), that is, as a factor group of (4.11) (with $r=3$). We may suppose l, m, n to be greater than 2, since $G^{m,n,p}$ has already been thoroughly investigated.

From $(bcd, cad, abd; 1)$ (a, b, c co-prime in pairs) we derive the

* Badoureau [1], p. 108, Fig. 76, incomplete.

group $((bcd, cad, abd; 2))$, of order $2abcd^2$,

from $(3, 3, 3; k)$, $((3, 3, 3; 2k))$, of order $18k^2$,
from $(3, 3, 12; 2)$, $((3, 3, 12; 4))$, of order 576,
from $(3, 3, 6; 2)$, $((3, 3, 6; 4))$, of order 288,
from $(3, 3, 4; 3)$, $((3, 3, 4; 6))$, of order 1008,
from $(3, 4, 4; 2)$, $((3, 4, 4; 4))$, of order 480,
from $(3, 4, 5; 2)$, $((3, 4, 5; 4))$, of order 720.*

We see also that $((3, 3, n; 4))$ collapses unless $n=2$, 3, 6, or 12, and that $((4, 4, 4; 4))$ and $((3, 3, 4; 8))$ are infinite.

The second method applies to the case when

$$b_1 = b_2 = \cdots = b_r \quad = q \text{ (say)}.$$

By adjoining an operator S, of period r, which cyclically permutes the generators $R_1, R_2, \cdots, R_r$, we derive the larger group

$$S^r = R_1^2 = (SR_1)^{rp} = (S^{-1}R_1SR_1)^q = 1.$$

Thus† *$((q^r; p))$ is an invariant subgroup of index r* in $(2, r, rp; q)$.

From $(2, 2, 2p; p) \sim [2p]$, we derive $((p, p; p)) \sim [p]$, of order $2p$,

from $(2, 3, 3; 2)$, the four-group $((2, 2, 2; 1))$,
from $(2, 3, 6k; 3)$, $((3, 3, 3; 2k))$ again,
from $(2, 3, 6; q)$, $((q, q, q; 2))$, of order $2q^2$,
from $(2, 4, 4; q)$, $((q, q, q, q; 1))$, of order $2q^2$,
from $(2, 4, 4p; 2)$, $((2, 2, 2, 2; p))$, of order $8p^2$,
from $(2, 5, 5; 2)$, $((2, 2, 2, 2, 2; 1))$, of order 16,

and from $(2, 3, 12; 4)$, the infinite group $((4, 4, 4; 4))$, again. We see also that the following are cases of collapse:‡

$$((q, q; p)) \quad (p \neq q), \qquad ((q, q, q; 1)) \quad (q \neq 2),$$

$$((2, 2, 2; p)) \ (p>2), \ ((3, 3, 3; p)) \ (p \text{ odd}), \text{ and } ((4, 4, 4; 3)).$$

Theorem C tells us that $((q, q, q, q; p))$ is infinite whenever $p>1$ or $q>2$, and that $((q^r; 1))$ is infinite when $r>4$, with the single exception of $((2, 2, 2, 2, 2; 1))$. When $r \leq 5$, $((2^r; 1))$ is the abelian group of order 2^{r-1} and type $(1, 1, \cdots, 1)$. This is obvious when $r<5$; however, when $r=5$ it is an in-

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* The method of §3.3, applied to the permutations (2.92), shows immediately that $((3, 4, 5; 4)) \sim G_{61/2} \times G_2$.

† Within the double parentheses, we use q^r to stand for $q, q, \cdots, q$.

‡ Enumerations, carried out independently by Sinkov and me (August–September, 1938), indicate the collapse of $((5, 5, 5; 3))$, and so also of $(2, 3, 9; 5)$ and $G^{3,9,10}$ (see Fig. 3). This is particularly interesting, as it shows that (3.81) is not a necessary condition for the collapse of $G^{m,n,p}$.

teresting consequence of our knowledge of the order of the group $(2, 5, 5; 2)$.

The b's being all equal (to q), the group picture described in §4.1 is now a regular map of r-gons, $2q$ at each vertex. Since the relation (4.121) identifies pairs of points which differ by rp steps along a Petrie polygon, this map is precisely $\{r, 2q\}_{rp}$. For instance, the sixteen pentagons of Fig. 8 represent the operators of the abelian group of order sixteen and type $(1, 1, 1, 1)$.

The third method applies to the case $r=3$, without restriction on the parity of p; but it is practical only when $p=3$ or 4.* By comparing (4.91) with

$$T_1^{k_1} = T_2^{k_2} = T_3^{k_3} = (T_1T_2)^2 = (T_2T_3)^2 = (T_1T_2T_3)^2 = 1,$$

which are Todd's† relations, we see that, for a suitable value of l,

$$((l, m, n; p)) \sim [n, p, m]',$$

where $[n, p, m]'$ denotes the rotation group of the regular polytope $\{n, p, m\}$ in four dimensions. Here l is the period of T_1T_3, and T_1T_3 is a displacement‡ which cyclically permutes the sides (and vertices) of a Petrie polygon. In other words, the Petrie polygon of $\{n, p, m\}$ has l sides.

By considering the particular polytopes in turn, we find§

$$((3, 3, 5; 3)) \sim [3, 3, 3]' \sim G_{5!/2},$$
$$((3, 4, 8; 3)) \sim [3, 3, 4]', \text{ of order } 192,$$
$$((3, 3, 12; 4)) \sim [3, 4, 3]', \text{ of order } 576,$$
$$((3, 5, 30; 3)) \sim [3, 3, 5]', \text{ of order } 7200.$$

Moreover, since $(T_1T_3)^{l/2}$ (l even) is the central inversion,‖ we have the central quotient groups

$$((3, 4, 4; 3)) \sim [3, 3, 4]'/G_2, \text{ of order } 96,$$
$$((3, 3, 6; 4)) \sim [3, 4, 3]'/G_2, \text{ of order } 288,$$
$$((3, 5, 15; 3)) \sim [3, 3, 5]'/G_2, \text{ of order } 3600,$$

which can be regarded as rotation groups in elliptic space.¶

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* Added in proof: By direct enumeration of cosets, J. M. Kingston found that $((3, 3, 4; 5)) \sim ((3, 3, 5; 5)) \sim G_{6!/2}$. As permutations, the generators are (2 3)(5 6), (1 2)(3 4), (1 2)(4 5) in the former, and (2 3)(4 5), (1 2)(3 4), (1 2)(4 6) in the latter.

† Todd [1], p. 217.

‡ $T_1T_3 = R_1R_2R_3R_4$; ibid., p. 224. See also Coxeter [2], p. 605, where the period of $R_1R_2R_3R_4$ is called h.

§ For the sake of uniformity, I have rearranged l, m, n into ascending order. For example, the first group would originally appear as $((5, 3, 3; 3))$. Although the meaning of the symbol $((l, m, n; p))$ is unchanged for all permutations of l, m, n, the meaning of $[n, p, m]'$ is unchanged only for transposition of m and n. Moreover, the two polytopes $\{n, p, m\}$, $\{m, p, n\}$, which have the same group, are not identical (unless $m=n$) but *reciprocal*.

‖ Coxeter [2], p. 606.

¶ Nos. XXVII, XXVIII, XXX of Goursat [1], pp. 67, 68.

We may also infer that $((3, 3, l; 3))$ collapses if $l \neq 5$, and that $((3, 4, l; 3))$ and $((3, 5, l; 3))$ collapse whenever l is not a divisor of 8, or 30, respectively.*

Each finite polytope $\{n, p, m\}$ determines a partition of the hypersphere into polyhedral *cells*, which form a three-dimensional "regular map." More generally, we use the same symbol $\{n, p, m\}$ to denote any regular map which fills a simply connected three-space (spherical, euclidean, or hyperbolic, according as $\sin \pi/m \sin \pi/n$ is greater than, equal to, or less than $\cos \pi/p$). The cells of this map are polyhedra $\{n, p\}$, m of which meet at each edge. A sequence of vertices P, Q, R, S, T, $\cdots$ is said to form a Petrie polygon of the map if P, Q, R, S are consecutive vertices of the Petrie polygon of one cell, Q, R, S, T of another, and so on.

Let $\{n, p, m\}_l$ denote the multiply connected map derived from $\{n, p, m\}$ by identifying certain pairs of edges,† namely, one pair which differ by l steps along a Petrie polygon, and every pair that results from this by applying a *displacement* (that is, an operation of the rotation group $[n, p, m]'$).

The group $((l, m, n; p))$, involving l, m, n symmetrically, is the intrinsic rotation group of any of the six regular maps

$$\{n, p, m\}_l, \quad \{m, p, n\}_l, \quad \{l, p, n\}_m, \quad \{n, p, l\}_m, \quad \{m, p, l\}_n, \quad \{l, p, m\}_n.$$

This includes our representation of $G^{m,n,p}$ on $\{n, p\}_m$ as a particular case, since the three-dimensional map $\{n, p, 2\}$ consists of just two cells, whose common boundary is the two-dimensional map $\{n, p\}$. The reflections of $[n, p]$ correspond to digonal rotations of $[n, p, 2]'$, just as the reflections of $[n]$ correspond to digonal rotations of $[n, 2]'$. The same principle was used by Todd in another connection.‡

The map $\{4, 3, 4\}$ is of special interest, since its space is euclidean. It is, in fact, the ordinary space-filling of cubes. Its Petrie polygon is permuted by a trigonal screw.§ Therefore the derived map $\{4, 3, 4\}_{3k}$ has k^3 times as many cells as $\{4, 3, 4\}_3$; that is, the order of $((3k, 4, 4; 3))$ is k^3 times that of $((3, 4, 4; 3))$. But we have seen that the latter, being also the rotation group of $\{3, 3, 4\}_4$, is of order 96. Hence $((4, 4, 3k; 3))$ is of order $96k^3$.

By (4.92), $G^{m,n,p}$ is the rotation group of any of the six maps

$$\{m, n, p\}_2, \quad \{m, p, n\}_2, \quad \{n, p, m\}_2, \quad \{n, m, p\}_2, \quad \{p, m, n\}_2, \quad \{p, n, m\}_2.$$

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* Added in proof: More precisely, $((3, 5, l; 3))$ collapses unless $l=3$, 5, 15, or 30. The group is icosahedral when $l=5$, as well as when $l=3$.

† Compare the definition of $\{m, n\}_p$ in §4.3. There we were able to say "all pairs," because we admitted reflections as well as displacements.

‡ Todd [1], p. 223.

§ Coxeter [5], p. 73.

The map $\{m, n, p\}_2$ has one or two vertices according as m is odd or even. Reciprocally, it has one or two cells according as p is odd or even. In particular, $\{3, 4, 3\}_2$, $\{4, 3, 3\}_2$, $\{3, 5, 5\}_2$, $\{5, 3, 5\}_2$ each consist of a single cell. The same holds for $\{5, 3, 3\}_3$, since $((3, 3, 5; 3)) \sim [5, 3]'$. In fact,

$$\{3, 4, 3\}_2, \ \{5, 3, 5\}_2, \ \{5, 3, 3\}_3$$

are easily recognized as the *octahedron-space** and the two *dodecahedron-spaces*.†

TABLE I

The known finite groups $(l, m \mid n, k)$

Group	*Order*	$2 \sin \pi/l \sin \pi/m - \cos \pi/n - \cos \pi/k$	*Where group is discussed*
$(2, 2 \mid k, k)$ } $\sim [k]$	$2k$	$4 \sin^2 \pi/2k$	1.1
$(2, k \mid 2, 2)$ }		$2 \sin \pi/k$	1.1
$(2i, 2j \mid 2, 2)$	$4ij$	$2 \sin \pi/2i \sin \pi/2j$	1.1
$(4, 4 \mid 2, k)$	$4k^2$	$2 \sin^2 \pi/2k$	1.3, 2.8
$(3, 3 \mid 3, k)$ }	$3k^2$	$2 \sin^2 \pi/2k$	1.6, 2.7, 2.8
$(3, k \mid 3, 3)$ }		$3^{1/2}(\sin \pi/k) - 1$	1.6, 1.7
$(3, 4 \mid 2, 4)$ }		0.518	1.5, 3.3
$(2, 4 \mid 3, 3)$ } $\sim G_{41}$	24	0.414	1.5
$(2, 3 \mid 4, 4)$ }		0.318	1.6
$(3, 5 \mid 2, 5)$ }		0.209	1.6
$(5, 5 \mid 2, 3)$ } $\sim G_{81/2} \sim \mathfrak{P}_1(5)$	60	0.191	1.1, 2.5, 3.7
$(2, 5 \mid 3, 3)$ }		0.176	1.6
$(2, 3 \mid 5, 5)$ }		0.114	1.6
$(4, 6 \mid 2, 3) \sim G_{81}$	120	0.207	1.3
$(4, 5 \mid 2, 4)$	160	0.124	1.4, 2.5, 3.3
$(4, 7 \mid 2, 3)$ }		0.114	1.6
$(3, 3 \mid 4, 4)$ } $\sim \mathfrak{P}_1(7)$	168	0.0858	1.6, 2.5, 3.5
$(3, 4 \mid 3, 4)$ }		0.0176	1.6, 2.7
$(4, 5 \mid 2, 5)$ } $\sim G_{61/2} \sim \mathfrak{P}_1(9)$	360	0.0223	1.6
$(5, 5 \mid 2, 4)$ }		−0.0161	1.6, 2.5, 3.5
$(3, 3 \mid 4, 5)$ }		−0.0161	1.7, 2.5
$(3, 4 \mid 3, 5)$ }	1080	−0.0843	1.7
$(3, 5 \mid 3, 4)$ }		−0.189	1.7, 3.7
$(6, 7 \mid 2, 3)$ } $\sim \mathfrak{P}_1(13)$	1092	−0.0661	1.7
$(7, 7 \mid 2, 3)$ }		−0.123	1.7, 2.5, 3.5
$(4, 8 \mid 2, 3)$	1152	+0.0412	1.3, 2.9
$(4, 9 \mid 2, 3) \sim \mathfrak{P}_1(17)$	2448	−0.0163	1.7

* Threlfall and Seifert [1], pp. 61–63.

† Weber and Seifert [1], pp. 242, 243. The hyperbolic dodecahedron-space has both two- and three-sided Petrie polygons (such as *ab* and *afb*); the spherical dodecahedron-space has both three- and five-sided Petrie polygons (such as *abg* and *figkh*). Since $((3, 5, 5; 3))$ is the icosahedral group, these two spaces are defined by the symbols $\{5, 3, 5\}_3$, $\{5, 3, 3\}_5$, just as well as by $\{5, 3, 5\}_2$, $\{5, 3, 3\}_3$, respectively. On the other hand, since $((3, 3, 3; 4))$ is of order 72, the map $\{3, 4, 3\}_3$ covers $\{3, 4, 3\}_2$ three times.

TABLE II
The known finite groups $(l, m, n; q)$

Group	*Order*	*Where group is discussed*
$(bcd, cad, abd; 1) \sim [abcd]' \times [d]'$*	$abcd^2$	2.2, 4.9
$(2, 2, n; n) \sim [n]$ (n odd)	$2n$	2.3
$(2, 2, 2q; q) \sim [2q]$	$4q$	2.3, 4.9
$(2, 4, 4; q)$	$8q^2$	2.5, 2.8, 4.9
$(2, 4, 2q; 2)$	$8q^2$	2.5, 2.8, 3.4
$(2, 3, 6; q)$	$6q^2$	2.5, 2.8, 4.9
$(2, 3, 2q; 3)$	$6q^2$	2.5, 2.8, 3.4
$(3, 3, 3; q)$	$9q^2$	2.7, 2.8, 4.9
$(2, 3, 3; 2) \sim G_{41/2}$	12	2.3, 3.2, 4.9
$(2, 3, 5; 5) \sim G_{51/2}$	60	2.3, 3.2
$(2, 5, 5; 2)$	80	2.5, 4.6, 4.9
$\left.\begin{matrix}(2, 4, 5; 3)\\(2, 5, 6; 2)\end{matrix}\right\} \sim G_{51}$	120	2.5, 4.6, 4.9
$(3, 3, 6; 2) \sim G_{41/2} \times G_{41/2}$	144	2.9, 4.9
$(2, 3, 7; 4) \sim \mathfrak{P}_1(7)$	168	2.3, 4.6
$(3, 4, 4; 2)$	240	2.9, 4.9
$(3, 3, 12; 2) \sim [3, 4, 3]''$	288	2.9, 4.9
$(2, 3, 8; 4) \sim \tilde{\mathfrak{P}}_1(7)$	336	2.5
$(3, 4, 5; 2) \sim G_{61/2}$	360	2.9, 4.9
$(3, 3, 4; 3) \sim \mathfrak{P}_1(7) \times G_3$	504	2.7, 4.9
$(2, 4, 5; 4) \sim G_{61/2} \times G_2$	720	2.5
$(2, 5, 8; 2)$	720	2.5, 3.5
$\left.\begin{matrix}(2, 3, 7; 6)\\(2, 3, 7; 7)\end{matrix}\right\} \sim \mathfrak{P}_1(13)$	1092	2.3
$\left.\begin{matrix}(2, 3, 8; 5)\\(2, 3, 10; 4)\end{matrix}\right\}$	2160	2.5, 3.5
$(2, 4, 7; 3) \sim \tilde{\mathfrak{P}}_1(13)$	2184	2.5
$(2, 6, 7; 2) \sim \mathfrak{P}_1(13) \times G_2$	2184	2.5
$(2, 5, 9; 2) \sim \mathfrak{P}_1(19)$	3420	2.6
$(2, 3, 11; 4) \sim \mathfrak{P}_1(23)$	6072	2.6

* a, b, c co-prime in pairs.

TABLE III
The known finite groups $G^{m,n,p}$

Group	*Order*	$\{m/n\}_p$ *Characteristic*	$\{m/n\}_p$ *Genus (when orientable)*	$1-\cos 2\pi/m$ $-\cos 2\pi/n-\cos 2\pi/p$	*Where group is discussed*
$G^{2,n,n} \sim [n] \times G_2$ (n even)	$4n$	-2	0	$4 \sin^2 \pi/n$	3.2
$G^{2,n,2n} \sim [n] \times G_2$ (n odd)	$4n$	-2	0	$2(1+4\cos^2 \pi/2n) \sin^2 \pi/2n$	3.2, 3.8
$G^{4,4,2q}$	$16q^2$	0	1	$\left.\right\}$ $2 \sin^2 \pi/2q$	3.4, 4.6
$G^{3,6,2q}$	$12q^2$	0	1*		
$G^{3,3,4} \sim G_{41}$	24	-2	0	2	3.2, 3.7, 4.8

* When $q=2$, we have $G^{3,4,6} \sim G_{41} \times G_2$. The genus of $\{3/4\}_6$ is, of course, zero.

TABLE III—*Continued*
The known finite groups $G^{m,n,p}$

Group	*Order*	$\{m/n\}_p$ *Characteristic*	$\{m/n\}_p$ *Genus (when orientable)*	$1-\cos 2\pi/m$ $-\cos 2\pi/n-\cos 2\pi/p$	*Where group is discussed*
$G^{3,5,5}\sim G_{51/2}$	60	−1		0.882	3.2, 3.9, 4.6
$G^{3,5,10}\sim G_{51/2}\times G_2$	120	−2	0	} 0.392	3.2, 4.8
$G^{4,5,5}\sim(4,5\mid 2,4)$	160	4		} 0.392	3.3, 3.8, 4.6
$G^{4,5,6}\sim G_{51}\times G_2$	240	6	4	0.191	3.5, 4.6
$G^{3,7,8}\sim\widetilde{\mathfrak{P}}_1(7)$	336	4	3	0.170	3.3, 3.9, 4.6
$G^{3,7,9}\sim\mathfrak{P}_1(8)$	504	6		0.111	3.7, 4.6
$G^{5,5,5}\sim\mathfrak{P}_1(11)$	660	33		0.0729	3.7
$G^{3,8,8}\sim\widetilde{\mathfrak{P}}_1(7)\times G_2$	672	14	8	0.0858	3.5
$G^{4,5,8}\sim(2,5,8;2)\times G_2$	1440	36	19	−0.0161	3.5
$G^{3,7,12}\sim\widetilde{\mathfrak{P}}_1(13)$	2184	26	14	+0.0105	3.3
$G^{3,7,13}$	2184	26		−0.00894	3.7, 3.9
$G^{3,7,14}\sim\widetilde{\mathfrak{P}}_1(13)$	2184	26	14	−0.0245	3.3
$G^{3,9,9}\sim\mathfrak{P}_1(19)$	3420	95		−0.0321	3.7
$G^{3,8,10}\sim(2,3,8;5)\times G_2$	4320	90	46	−0.0161	3.5
$G^{4,6,7}\sim\widetilde{\mathfrak{P}}_1(13)\times G_2$	4368	182		−0.123	3.5
$G^{4,5,9}\sim\widetilde{\mathfrak{P}}_1(19)$	6840	171		−0.0751	3.3
$G^{3,8,11}\sim\widetilde{\mathfrak{P}}_1(23)$	12144	253		−0.0483	3.3
$G^{3,7,15}\sim\mathfrak{P}_1(29)$	12180	145		−0.0370	3.7

APPENDIX. ON THE COLLAPSE OF (2, 3, 9; 4) AND (2, 5, 7; 2)*

(2, 3, 9; 4): In the form (3.65), this group is defined by $P^9=Q^4=(P^2Q)^3=(P^3Q)^2=1$. Since $QP^3Q=P^{-3}$, we have $(P^3Q^2)^3=1$. Similarly,† $(PQP^2Q^{-1})^2=1$, $(P^4Q)^3=1$, $(P^2Q^2)^3=1$. Using these relations, we find that

$$\begin{aligned}
P^5Q^2P^4Q^2 &= P^7\cdot P^{-2}Q^{-2}P^{-2}\cdot P^{-3}Q^{-1}\cdot Q^{-1} = P^7\cdot Q^2P^2Q^2\cdot QP^3\cdot Q^{-1}\\
&= P^7Q\cdot QP^2Q^{-1}P\cdot P^2Q^{-1} = P^7Q\cdot P^{-1}QP^{-2}Q^{-1}\cdot P^2Q^{-1}\\
&= P^3(P^4QP^4)^2P^3Q\cdot Q^2P^2Q^2\cdot Q\\
&= P^3(Q^{-1}P^{-4}Q^{-1})^2Q^{-1}P^{-3}\cdot P^{-2}Q^{-2}P^{-2}\cdot Q\\
&= P^3Q^{-1}P^{-2}(P^{-2}Q^{-2}P^{-2})^2P^{-3}Q^{-2}P^{-3}\cdot PQ\\
&= P^3Q^{-1}P^{-2}(Q^2P^2Q^2)^2Q^2P^3Q^2\cdot PQ\\
&= P^3Q^{-1}(P^{-2}Q^{-2})^2PQ = P^3Q^{-1}\cdot Q^2P^2\cdot PQ\\
&= (P^3Q)^2 = 1.
\end{aligned}$$

Hence

* For this Appendix I am indebted to Dr. A. Sinkov. (See §2.6.)

† Sinkov [3], pp. 68, 69, (2), (4), (5).

$$P^3 = (P^{-3})^2 = (P^{-3}\cdot P^5Q^2P^4Q^2)^2$$
$$= (P^2Q^2P^2\cdot P^2Q^2)^2 = (Q^{-2}P^{-2}Q^{-2}\cdot P^2Q^2)^2 = 1.$$

Thus (2, 3, 9; 4) collapses to (2, 3, 3; 2), which is the tetrahedral group.

(2, 5, 7; 2): In the form (2.12), this group is defined by $R^5=S^7=(RS)^2=(R^2S^2)^2=1$, whence

$$S^3\cdot R^3SR^3S^2R\cdot S^3 = S\cdot S^2R^2\cdot RSR\cdot R^2S^2\cdot RS^3 = S\cdot R^{-2}S^{-2}\cdot S^{-1}\cdot S^{-2}R^{-2}\cdot RS^3$$
$$= SR\cdot R^2S^2R^2\cdot R^2S^2\cdot S = R^{-1}S^{-1}\cdot S^{-2}\cdot S^{-2}R^{-2}\cdot S$$
$$= R^2\cdot R^2S^2R^2\cdot RS = R^2\cdot S^{-2}\cdot S^{-1}R^{-1}$$
$$= R^3\cdot R^{-1}S^{-1}\cdot S^{-2}R^{-2}\cdot R = R^3\cdot SR\cdot R^2S^2\cdot R$$
$$= R^3SR^3S^2R.$$

It follows that

$$S\cdot R^3SR^3S^2R\cdot S = R^3SR^3S^2R,\quad SR^2\cdot RSR\cdot R^2S\cdot SRS = R^2\cdot RSR\cdot R^2S^2\cdot R,$$
$$SR^2\cdot S^{-1}\cdot R^2S\cdot R^{-1} = R^2\cdot S^{-1}\cdot S^{-2}R^{-2}\cdot R,$$

so that

$$SR^2S^{-1} = R^2S^{-3}R^{-1}\cdot RS^{-1}R^{-2} = R^2S^3R^{-2}.$$

But SR^2S^{-1} is of period five, while $R^2S^3R^{-2}$ is of period seven. Hence $R=S=1$, and (2, 5, 7; 2) collapses totally.

References

Badoureau, 1. *Mémoire sur les figures isoscèles*, Journal de l École Polytechnique, vol. 30 (1881), pp. 47–172.

Brahana, 1. *Regular maps and their groups*, American Journal of Mathematics, vol. 49 (1927), pp. 268–284.

———, 2. *Certain perfect groups generated by two operators of orders two and three*, American Journal of Mathematics, vol. 50 (1928), pp. 345–356.

———, 3. *Pairs of generators of the known simple groups whose orders are less than one million*, Annals of Mathematics, (2), vol. 31 (1930), pp. 529–549.

———, 4. *On the groups generated by two operators of orders two and three whose product is of order eight*, American Journal of Mathematics, vol. 53 (1931), pp. 891–901.

Brahana and Coble, 1. *Maps of twelve countries with five sides with a group of order* 120 *containing an icosahedral subgroup*, American Journal of Mathematics, vol. 48 (1926), pp. 1–20.

Burnside, 1. *Theory of Groups of Finite Order*, Cambridge, 1911.

Carmichael, 1. *Abstract definitions of the symmetric and alternating groups and certain other permutation groups*, Quarterly Journal of Mathematics, vol. 49 (1923), pp. 226–283.

Coxeter, 1. *The polytopes with regular-prismatic vertex figures*, Proceedings of the London Mathematical Society, (2), vol. 34 (1932), pp. 126–189.

———, 2. *Discrete groups generated by reflections*, Annals of Mathematics, (2), vol. 35 (1934), pp. 588–621.

———, 3. *Abstract groups of the form* $V_i^k=V_j^3=(V_iV_j)^2=1$, Journal of the London Mathematical Society, vol. 9 (1934), pp. 213–219.

———, 4. *The complete enumeration of finite groups of the form* $R_i^2=(R_iR_j)^{k_{ij}}=1$, Journal of the London Mathematical Society, vol. 10 (1935), pp. 21–25.

———, 5. *The groups determined by the relations* $S^l=T^m=(S^{-1}T^{-1}ST)^p=1$, Duke Mathematical Journal, vol. 2 (1936), pp. 61–73.

———, 6. *The abstract groups* $R^m = S^m = (R^jS^j)^{p_j} = 1$, $S^m = T^2 = (S^jT)^{2p_j} = 1$ *and* $S^m = T^2 = (S^{-j}TS^jT)^{p_j}=1$, Proceedings of the London Mathematical Society, (2), vol. 41 (1936), pp. 278–301.

———, 7. *Regular skew polyhedra in three and four dimensions, and their topological analogues*, Proceedings of the London Mathematical Society, (2), vol. 43 (1937), pp. 33–62.

Dickson, 1. *Linear Groups*, Leipzig, 1901.

Dyck, 1. *Gruppentheoretische Studien*, Mathematische Annalen, vol. 20 (1882), pp. 1–44.

Edington, 1. *Abstract group definitions and applications*, these Transactions, vol. 25 (1923), pp. 193–210.

Goursat, 1. *Sur les substitutions orthogonales et les divisions régulières de l'espace*, Annales Scientifique de l'École Normale Supérieure, (3), vol. 6 (1889), pp. 9–102.

Hamilton, 1. *Memorandum respecting a new system of roots of unity*, Philosophical Magazine, (4), vol. 12 (1856), p. 446.

Klein, 1. *Vorlesungen über das Ikosaeder*, Leipzig, 1884.

Klein-Fricke, 1, 2. *Vorlesungen über die Theorie der elliptischen Modulfunktionen*, Leipzig, 1890.

Lewis, 1. *Note on the defining relations for the simple group of order* 660, Bulletin of the American Mathematical Society, vol. 44 (1938), p. 456.

Miller, 1. *Groups generated by two operators of order three whose product is of order four*, Bulletin of the American Mathematical Society, vol. 26 (1920), pp. 361–369.

———, 2. *The commutator subgroup of a group generated by two operators*, Proceedings of the National Academy of Sciences, vol. 18 (1932), pp. 665–668.

Schreier and van der Waerden, 1. *Die Automorphismen der projektiven Gruppen*, Abhandlungen aus dem Mathematischen Seminar, Hamburg, vol. 6 (1928), pp. 203–322.

Sinkov, 1. *A set of defining relations for the simple group of order* 1092, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 237–240.

———, 2. *The groups determined by the relations* $S^l=T^m=(S^{-1}T^{-1}ST)^p=1$, Duke Mathematical Journal, vol. 2 (1936), pp. 74–83.

———, 3. *Necessary and sufficient conditions for generating certain simple groups by two operators of periods two and three*, American Journal of Mathematics, vol. 59 (1937), pp. 67–76.

———, 4. *Notes on the groups of genus one*, Tôhoku Mathematical Journal, vol. 43 (1937), pp. 164–170.

———, 5. *On the group-defining relations* (2, 3, 7; p), Annals of Mathematics, (2), vol. 38 (1937), pp. 577–584.

———, 6. *On generating the simple group* $LF(2, 2^N)$ *by two operators of periods two and three*, Bulletin of the American Mathematical Society, vol. 44 (1938), pp. 449–455.

Threlfall, 1. *Gruppenbilder*, Leipzig, 1932.

Threlfall and Seifert, 1. *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Mathematische Annalen, vol. 104 (1931), pp. 1–70.

Todd, 1. *The groups of symmetries of the regular polytopes*, Proceedings of the Cambridge Philosophical Society, vol. 27 (1931), pp. 212–231.

———, 2. *A note on the linear fractional group*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 195–200.

Todd and Coxeter, 1. *A practical method for enumerating co-sets of a finite abstract group*, Proceedings of the Edinburgh Mathematical Society, (2), vol. 5 (1936), pp. 26–34.

Weber and Seifert, 1. *Die beiden Dodekaederräume*, Mathematische Zeitschrift, vol. 37 (1933), pp. 237–253.

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